MATH 227: Introduction to Complex Analysis

Spring 2017-2018, Midterm 1, Duration: 60 min. + 15 min.

Name:________________________ AUB ID:________________________

Exercise	Points	Scores
1	25	
2	25	
3	25	
4	25	
Total	100	

INSTRUCTIONS:

(a) Explain your answers precisely and clearly to ensure full credit.

(b) No book. No notes. No calculators.

Exercise 1. For any $z \in \mathbb{C}$, define the hyperbolic cosine and sine functions as

$$\cosh z := \frac{e^z + e^{-z}}{2}, \qquad \sinh z := \frac{e^z - e^{-z}}{2}.$$

(a) **(2 points)** Show that $\cosh(iz) = \cos(z)$ and $\sinh(iz) = i\sin(z)$.

We have

$$\cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\sin(z)$$

(b) **(3 points)** Find the sets of points for which $\cosh z = 0$ and $\sinh z = 0$.

$$\begin{aligned} \cosh z = 0 &\iff e^{z} + e^{-z} = 0 \\ &\iff e^{2z} = -1 \\ &\iff 2z = (2k+1)\pi i \qquad k \in \mathbb{Z} \\ &\iff z = \frac{2k+1}{2}\pi i \qquad k \in \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \sinh z = 0 &\iff e^{z} - e^{-z} = 0 \\ &\iff e^{2z} = 1 \\ &\iff 2z = 2k\pi i \qquad k \in \mathbb{Z} \\ &\iff z = k\pi i \qquad k \in \mathbb{Z} \end{aligned}$$

(c) **(4 points)** Compute the power series of $\cosh z$ and $\sinh z$ around the origin. (You may assume the power series of $\exp z$.)

Recall, for all $z \in \mathbb{C}$, $e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$

So

$$2\cosh z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k + \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k z^k = \sum_{k=0}^{\infty} \frac{1}{k!}(1 + (-1)^k) z^k$$

$$\text{But } 1 + (-1)^k = \begin{cases} 2 & \text{if } k = 2\ell \quad \ell \in \mathbb{Z} \\ 0 & \text{if } k = 2\ell+1 \quad \ell \in \mathbb{Z} \end{cases}$$

$$\therefore \cosh z = \sum_{\ell=0}^{\infty} \frac{1}{(2\ell)!} z^{2\ell}$$

Similarly

$$2\sinh z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k - \sum_{k=0}^{\infty} \frac{1}{k!}(-1)^k z^k = \sum_{k=0}^{\infty} \frac{1}{k!}(1 - (-1)^k) z^k$$

$$\text{But } 1 - (-1)^k = \begin{cases} 0 & \text{if } k = 2\ell \quad \ell \in \mathbb{Z} \\ 2 & \text{if } k = 2\ell+1 \quad \ell \in \mathbb{Z} \end{cases}$$

$$\therefore \sinh z = \sum_{\ell=0}^{\infty} \frac{1}{(2\ell+1)!} z^{2\ell+1}$$

(d) **(5 points)** Compute $(\cosh z)'$ and $(\sinh z)'$.

$$(\cosh z)' = \left(\frac{e^{z}+e^{-z}}{2}\right)' = \frac{e^{z}-e^{-z}}{2} = \sinh z$$

$$(\sinh z)' = \left(\frac{e^{z}-e^{-z}}{2}\right) = \frac{e^{z}+e^{-z}}{2} = \cosh z$$

(e) **(6 points)** Prove that $\cosh(z+w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$ and $\sinh(z+w) = \cosh(z)\sinh(w) + \sinh(z)\cosh(w)$.

$$\cosh(z+\omega) = \frac{1}{2}\left(e^{z+\omega} + e^{-z-\omega}\right) \qquad \sinh(z+\omega) = \frac{1}{2}\left(e^{z+\omega} - e^{-z-\omega}\right)$$

$$\cosh z \cosh\omega = \frac{1}{4}\left(e^{z}+e^{-z}\right)\left(e^{\omega}+e^{-\omega}\right) = \frac{1}{4}\left(e^{z+\omega}+e^{z-\omega}+e^{-z+\omega}+e^{-z-\omega}\right) \quad \text{— ①}$$

$$\sinh z \sinh\omega = \frac{1}{4}\left(e^{z}-e^{-z}\right)\left(e^{\omega}-e^{-\omega}\right) = \frac{1}{4}\left(e^{z+\omega}-e^{z-\omega}-e^{-z+\omega}+e^{-z-\omega}\right) \quad \text{— ②}$$

$$\cosh z \sinh\omega = \frac{1}{4}\left(e^{z}+e^{-z}\right)\left(e^{\omega}-e^{-\omega}\right) = \frac{1}{4}\left(e^{z+\omega}-e^{z-\omega}+e^{-z+\omega}-e^{-z-\omega}\right) \quad \text{— ③}$$

① + ② gives $\cosh(z)\cosh(\omega) + \sinh(z)\sinh(\omega) = \cosh(z+\omega)$

③ + ③$(z \leftrightarrow \omega)$ gives $\sinh(z+\omega) = \cosh(z)\sinh(\omega) + \sinh(z)\cosh(\omega)$.

(f) **(5 points)** Show that

$$|\cosh(z)|^2 = \cosh^2(\Re(z))\cos^2(\Im(z)) + \sinh^2(\Re(z))\sin^2(\Im(z)),$$
$$|\sinh(z)|^2 = \sinh^2(\Re(z))\cos^2(\Im(z)) + \cosh^2(\Re(z))\sin^2(\Im(z)).$$

Note $\cosh(z) = \cosh(x+iy) = \cosh(x)\cosh(iy) + \sinh(x)\sinh(iy)$ by (e)

$z = x+iy$, $x,y \in \mathbb{R}$

$= \cosh(x)\cos(y) + i\sinh(x)\sin(y)$ by (a)

$\therefore \; |\cosh(z)|^2 = (\cosh z\, \overline{\cosh z}) = \cosh^2(x)\cos^2(y) + \sinh^2(x)\sin^2(y)$

Similarly,

$\sinh(z) = \sinh(x+iy) = \sinh(x)\cosh(iy) + \cosh(x)\sinh(iy)$

$= \sinh(x)\cos(y) + i\cosh(x)\sin(y)$

$\therefore \; |\sinh(z)| = \sinh^2(x)\cos^2(y) + \cosh^2(x)\sin^2(y)$

Exercise 2.(25 points) Let $f = u + iv$ be a complex-valued function on some open connected subset of $\mathbb{C}$ where u and v are real-valued functions.

(a) **(5 points)** State the Cauchy-Riemann equations.

Bookwork

(b) **(20 points)** Show that if f is holomorphic at a point, then u and v satisfy the Cauchy-Riemann equations at that point.

Bookwork

Exercise 3.

(a) Define $u : \mathbb{C} \to \mathbb{R}$ by $u(x,y) = a_4x^4 + 4a_3x^3y + 6a_2x^2y^2 + 4a_1xy^3 + a_0y^4$ for some constants $a_k \in \mathbb{C}$.

i. **(10 points)** Find necessary conditions on the a_k for u to be harmonic.

We compute

$$u_x = 4a_4x^3 + 12a_3x^2y + 12a_2xy^2 + 4a_1y^3$$
$$u_{xx} = 12a_4x^2 + 24a_3xy + 12a_2y^2$$
$$u_y = 4a_3x^3 + 12a_2x^2y + 12a_1xy^2 + 4a_0y^3$$
$$u_{yy} = 12a_2x^2 + 24a_1xy + 12a_0y^2$$

u harmonic $\iff u_{xx} + u_{yy} = 0 \iff 12(a_4+a_2)x^2 + 24(a_3+a_1)xy + 12(a_2+a_0)y^2$

$\iff \boxed{a_4 = -a_2 = a_0 \text{ and } a_3 = -a_1}$

Hence

$u(x,y) = a_4(x^4 - 6x^2y^2 + y^4) + 4a_3(x^3y - xy^3)$ is harmonic.

ii. **(10 points)** Find a holomorphic function f such that $\Re(f) = u$ where u is given as in i).

We seek a differentiable function v s.t the Cauchy-Riemann equations hold with u found in i) $v_y = u_x$ and $v_x = -u_y$

Now,

$$v_y = u_x = 4a_4(x^3 - 3xy^2) + 4a_3(3x^2y - y^3)$$
$$\Rightarrow v = 4a_4(x^3y - xy^3) + a_3(6x^2y^2 - y^4) + F(x) \quad \text{for some function } F.$$

Next,

$$v_x = -u_y \iff 4a_4(3x^2y - y^3) + 12a_3xy^2 + F'(x) = -4a_3(x^3 - 3xy^2) + 4a_4(3x^2y - y^3)$$
$$\iff F'(x) = -4a_3x^3$$
$$\iff F(x) = -a_3x^4 + K \quad \text{for some constant } K$$

Hence,

$$v(x,y) = 4a_4(x^3y - xy^3) - a_3(x^4 - 6x^2y^2 + y^4) + K$$

Since u_x, u_y, v_x and v_y are continuous on $\mathbb{C}$, the complex-valued function

$$f(x,y) := u(x,y) + iv(x,y) = (a_4 - ia_3)(x^4 + 4ix^3y - 6x^2y^2 - 4ixy^3 + y^4) + iK$$
$$= Az^4 + iK \quad \text{where } A = a_4 - ia_3$$

is holomorphic on $\mathbb{C}$.

(b) **(5 points)** Let $u : \mathbb{C} \to \mathbb{R}$ be defined by $u(x, y) = \cos(xy)$. Is there a holomorphic function f such that $\Re f = u$?

Suppose there exists a holomorphic function f s.t. $\Re f = u$. Then u and $v = \Im f$ satisfy the Cauchy-Riemann equations. In particular, both u and v must be harmonic.

Now

$$u_x = -y\sin(xy)$$

$$u_{xx} = -y^2\cos(xy)$$

$$u_y = -x\sin(xy)$$

$$u_{yy} = -x^2\cos(xy)$$

so that

$$u_{xx} + v_{yy} = -(x^2+y^2)\cos(xy)$$

So $u_{xx} + v_{yy} = 0$ at $(x,y) = (0,0)$ and on curves $xy = \frac{\pi}{2} + k\pi \quad k \in \mathbb{Z}$.

But these sets are not open in the topology on $\mathbb{C}$.

Hence there is no holomorphic function f s.t. $\mathrm{Re}(f) = u$.

Exercise 4.

(a) (**10 points**) Let $z_0 \in \mathbb{C}$ and $k \in \mathbb{Z}$. Compute the contour integral $\oint_\gamma \cancel{z^k} \mathrm{dz}$, $\oint_\gamma (z-z_0)^k dz$
where γ is the positively oriented circle of radius r_0 centred at z_0.

Bookwork (use parametrisation and compute explicitly)

(b) (**5 points**) Compute $\oint_\gamma f(z)\mathrm{d}z$ where $f(z) = \sum_{n=-k}^{\ell} a_n z^n$ and $k, \ell \in \mathbb{N}$ and a_j are complex constants for $j = -k, \ldots, \ell$, and γ is any positively oriented closed curve around the origin.

We have

$$\oint_\gamma f(z)\,dz = \oint_\gamma \sum_{n=-k}^{-2} a_n z^n\,dz + \oint_\gamma a_{-1} z^{-1} dz + \oint_\gamma \sum_{n=0}^{\ell} a_n z^n\,dz$$

$$= \oint_{|z|=1} \sum_{n=-k}^{-2} a_n z^n\,dz + \oint_{|z|=1} a_{-1} z^{-1} dz + \oint_{|z|=1} \sum_{n=0}^{\ell} a_n z^n\,dz$$

by deformation to the unit circle centred 0 taken to be positively oriented

$$= 0 + 2\pi i a_{-1} + 0$$

$$= 2\pi i a_{-1}$$

by (a)

(c) **(10 points)** Integrate the function $f(z) = \dfrac{1}{z^4 - z^3 + z^2}$ along the positively oriented parallelogram with vertices $(-\frac{1}{2}, -\frac{1}{2})$, $(1,0)$, $(1,2)$ and $(-\frac{1}{2}, \frac{3}{2})$.

We compute

$$z^4 - z^3 + z^2 = z^2(z^2 - z + 1) = z^2\left(\left(z - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right) = z^2\left(z - \tfrac{1+\sqrt{3}i}{2}\right)\left(z - \tfrac{1-\sqrt{3}i}{2}\right)$$

$$= z^2(z-a)(z-\bar{a}) \quad \text{where } a = e^{\frac{i\pi}{3}} = \frac{1+\sqrt{3}i}{2}$$

So $f(z)$ has singularities at $0, e^{\frac{i\pi}{3}}, e^{-\frac{i\pi}{3}}$.

Note $a\bar{a} = 1$, $\quad a + \bar{a} = 2\cos\frac{\pi}{3} = 1$, $\quad a - \bar{a} = 2i\sin\frac{\pi}{3} = \sqrt{3}\,i$

Use partial fraction decomposition

$$\frac{1}{z^2(z-a)(z-\bar{a})} = \frac{A}{z^2} + \frac{B}{z} + \frac{C}{z-a} + \frac{D}{z-\bar{a}} \qquad \text{for some } A, B, C, D \text{ to determine.}$$

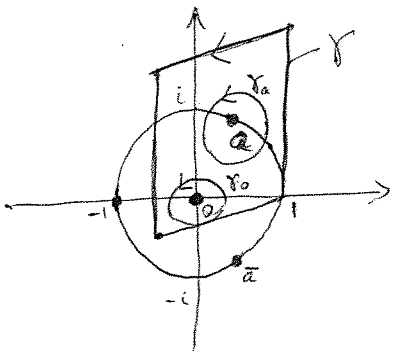
$$\Rightarrow \quad 1 = A(z-a)(z-\bar{a}) + Bz(z-a)(z-\bar{a}) + Cz^2(z-\bar{a}) + Dz^2(z-a)$$

Evaluate at $z = 0 \quad \Rightarrow \quad 1 = A a\bar{a} \quad \Rightarrow \boxed{A = 1}$

$z = a \qquad 1 = Ca^2(a - \bar{a}) \quad \Rightarrow \quad C = \dfrac{e^{-\frac{2\pi}{3}i}}{\sqrt{3}i} = \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\frac{1}{\sqrt{3}i} = \frac{-1}{2} + \frac{i}{2\sqrt{3}}$

$z = \bar{a} \qquad 1 = D\bar{a}^2(\bar{a} - a) \quad \Rightarrow \quad D = \bar{C}$

For B, we get from the z-term, $\quad 0 = A(-a-\bar{a}) + Ba\bar{a} \quad \Rightarrow \boxed{B = 1}$



Hence,

$$\oint_\gamma f(z)dz = \int_\gamma \frac{A}{z^2}dz + \int_\gamma \frac{B}{z}dz + \int_\gamma \frac{C}{z-a}dz + \int_\gamma \frac{D}{z-\bar{a}}dz$$

Since γ does not enclose $\bar{a}$, we have $\displaystyle\int_\gamma \frac{D}{z-\bar{a}}dz = 0$ by Cauchy Theorem

By deformation, $\displaystyle\int_\gamma \frac{A}{z^2}dz = \int_{\gamma_0} \frac{A}{z^2}dz = 0$

$$\int_\gamma \frac{B}{z}dz = \int_{\gamma_0} \frac{B}{z}dz = B2\pi i$$

$$\int_\gamma \frac{C}{z-a}dz = \int_{\gamma_a} \frac{C}{z-a}dz = C2\pi i$$

by (a) above and where γ_0 is a small circle around 0 and γ_a small circle around a (all taken clockwise)

Hence

$$\oint_\gamma f(z)dz = 2\pi i(B + C) = 2\pi i\left(-\frac{1}{2} + \frac{i}{2\sqrt{3}}\right) = \boxed{\pi\left(\frac{-1}{\sqrt{3}} + i\right)}$$