Math 227–Final Exam (Spring 00)

B. Shayya

1. (30 pts) Evaluate

$$\int_0^{2\pi} \frac{1}{5+4\cos\theta}\, d\theta$$

2. Let

$$F(z) = \frac{\log z}{1+z^3}.$$

(a) (10 pts) Show that

$$\mathrm{Res}\left(F, e^{i\pi/3}\right) = \frac{1}{2\pi i}\left(\frac{\pi^2}{9} + i\frac{\pi^2}{3\sqrt{3}}\right).$$

(b) (15 pts) Find $\lim_{\rho\to 0} I_\rho$ and $\lim_{R\to\infty} I_R$, where

$$I_r = \int_0^{2\pi/3} F\left(re^{i\theta}\right) ire^{i\theta}\, d\theta \qquad (r > 0,\ \ r \neq 1.)$$

(c) (5 pts) Show that for $x > 0$,

$$F(x) - e^{2\pi i/3}F\left(e^{2\pi i/3}x\right) = \frac{3-i\sqrt{3}}{2}F(x) + \frac{\pi}{3}\frac{\sqrt{3}+i}{1+x^3}. \qquad (*)$$

(d) (20 pts) Evaluate the improper integral

$$\int_0^\infty \frac{\ln x}{1+x^3}\, dx.$$

(e) (5 pts) Find all $z \in \mathbb{C}$ for which (∗) (with x replaced by z) holds.

3. (22 pts) Show that $\sqrt{3z^2-1}$ can be defined to be a single valued holomorphic function for $|z| > 1$, and that $\sqrt[4]{5-z}$ can be defined to be a single valued holomorphic function for $|z| < 2$.

4. (33 pts) Let $U = \{z = x+iy : y > 0\}$ be the upper half plane. Let $f \in O(U)$ and suppose $\lim_{z\to 0, z\in U} f(z) = 0$. Show that

$$\lim_{z\to 0, z\in U} zf'(z) = 0$$

uniformly on every angle $|\theta - \frac{\pi}{2}| \leq \alpha < \pi/2$.

5. Let $f \in O(\mathbb{C} - 0)$ and suppose $|f(z)| = 1$ for $|z| = 1$.

(a) (12 pts) For $z \neq 0$, define $g(z) = \overline{f(1/\overline{z})}$. Prove that $g \in O(\mathbb{C} - 0)$.
(b) (12 pts) Prove that $f(z) \neq 0$ for $z \neq 0$.
(c) (12 pts) Show that $\mathrm{Res}(f'/f, 0) \in \mathbb{Z}$.
(d) (24 pts) Prove that there exist an integer m and a function $\phi \in O(\mathbb{C} - 0)$ such that

$$f(z) = z^m e^{\phi(z)}, \qquad z \neq 0.$$

Also prove that $\phi(z) + \overline{\phi(1/\overline{z})} = 0$ for $z \neq 0$.