Math 227–Final Exam (Spring 01) B. Shayya

1. (32 pts) Evaluate

$$(a) \quad \int_{[-1-i,3+i\sqrt{3}]} \frac{dz}{z} \qquad \text{and} \qquad (b) \quad \int_{|z-i|=1} \frac{dz}{(z-i)^2(z+1)}.$$

2. (32 pts) Let $x \in \mathbb{R}$ and $y > 0$. Find

$$\sum_{n=0}^{\infty} e^{-ny} \cos nx.$$

(Hint: $\operatorname{Re} e^{inz}$.)

3. (32 pts) Show that all zeroes of $p(z) = 3z^3 - 2z^2 + 2iz - 8$ lie in the ring $1 < |z| < 2$.

4. (32 pts) Let $f(z)$ be holomorphic in the ring $A < |z| < B$ and suppose $|f(z)| = |z|^\mu$ there. Show that μ is an integer.

5. Let $0 \le R < 1$, $\lambda > 0$, and $\Omega = \mathbb{C} - ((-\infty, -1] \cup [1, \infty))$. For $z \in \Omega$, define

$$F(z) = \frac{e^{\lambda i z^2}}{\sqrt{1 - z^2}}.$$

(a) (16 pts) Prove that

$$z \in \Omega \implies 1 - z^2 \in \mathbb{C} - (-\infty, 0],$$

so that $F \in \mathcal{O}(\Omega)$.

(b) (16 pts) Prove that

$$\left|\sqrt{1 - it^2}\right| = (1 + t^4)^{1/4} \ge 1 \qquad \text{and} \qquad \left|\sqrt{1 - R^2 e^{2i\theta}}\right| \ge \sqrt{1 - R^2}$$

for $t, \theta \in \mathbb{R}$.

(c) (25 pts) Prove that

$$\int_0^R \left|F\left(te^{i\pi/4}\right)\right| dt \le \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{\lambda}} \qquad \text{and} \qquad \int_0^{\pi/4} \left|F\left(Re^{i\theta}\right)\right| d\theta \le \frac{\pi}{4R^2\lambda\sqrt{1 - R^2}}.$$

(Hint: $\int_0^\infty e^{-u^2} du = \sqrt{\pi}/2$.)

(d) (15 pts) Prove that

$$\left|\int_0^R F(x)dx\right| \le \frac{1}{\sqrt{\lambda}} \left(\frac{\sqrt{\pi}}{2} + \frac{\pi}{4R\sqrt{\lambda}\sqrt{1 - R^2}}\right).$$