Math 227–Final Exam (Spring 02)

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Problem 1

Let $f(z)$ be holomorphic in the ring $A < |z| < B$ and suppose $|f(z)| = |z|^\mu$ there.

(a) (10 pts) Find the logarithmic derivative of f.

(b) (5 pts) Show that μ is an integer.

Problem 2

(25 pts) Let

$$f(z) = \sum_{n=1}^{\infty} A_n z^n$$

be holomorphic in the open disc $|z| < 1$. Also, let the point $z = 1$ be an isolated, but not an essential singularity of $f(z)$. What, beyond not being essential, is true of the singularity if $|A_n| \leq n$?

Problem 3

Let $\varphi(z)$ be harmonic in $\mathbb{C} - 0$.

(a) (15 pts) Prove that

$$\varphi(z) = \text{Re} f(z) + A \log |z|$$

with $f(z)$ holomorphic in $\mathbb{C} - 0$.

(b) (10 pts) Is A unique ?

Problem 4

Let h be an entire function such that $|h(z)| \geq 11$ on the circle $|z| = 5$.

(a) (15 pts) Prove that the two equations

$$h(z) = 4 + 3i \qquad \text{and} \qquad h(z) = 8$$

have the same number of solutions in the disc $|z| \leq 5$.

(b) (5 pts) Prove that

$$\sup_{|z|=6} |h(z)| \geq 10.$$

Problem 5

(15 pts) Let $p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_0$, $n \geq 1$. Prove that

$$\sup_{z \in \overline{D}} |p(z)| \geq 1.$$

where $\overline{D}$ is the closed unit disc.