Math 227 – Final Exam (Spring 04) B. Shayya

Problem 1

(9 pts each) Use complex integration to evaluate the following integrals.

$$\text{(i)} \int_0^{\infty} \frac{1}{1+x^4}\,dx \qquad \text{(ii)} \int_0^{\pi} \frac{dx}{5-4\cos x}$$

Problem 2

(a) (12 pts) Prove that if u is harmonic in a star-shaped set X, then $u = \operatorname{Re} f$ for some $f \in O(X)$.

(b) (5 pts) State and prove Liouville's theorem for harmonic functions.

(c) (15 pts) Let φ be harmonic in the punctured plane $z \neq 0$. If φ is positive there, prove that φ is a constant.

Problem 3

Let

$$f(z) = \sum_{n=0}^{\infty} A_n z^n$$

be holomorphic in the open disc $|z| < 1$. Also, let the point $z = 1$ be an isolated, but not an essential singularity of $f(z)$.

(a) (17 pts) What, beyond not being essential, is true of the singularity if $A_n \to 0$ as $n \to \infty$?

(b) (8 pts) What if $|A_n| \le 1$ for all n ?

Problem 4

Suppose $f(z) = u(z) + iv(z)$ is an entire function with $v(0) = 0$.

(a) (9 pts) Write $f(z) = A_0 + \sum_{n=1}^{\infty} 2A_n z^n$. Show that if $R > 0$, then

$$A_n = \frac{1}{2\pi R^n} \int_0^{2\pi} u(Re^{i\theta}) e^{-in\theta}\, d\theta \qquad (n = 0, 1, \ldots).$$

(b) (6 pts) Show that if $|z| < R$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{Re^{i\theta} + z}{Re^{i\theta} - z}\, u(Re^{i\theta})\, d\theta.$$

(c) (10 pts) Suppose there exist $A, B, \alpha > 0$ such that

$$|u(z)| \le A + B|z|^{\alpha}.$$

Show that there exist $C, D > 0$ such that

$$|f(z)| \le C + D|z|^{\alpha}.$$