Math 224 – Final Exam (Fall 03)

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Problem 1 True or False:

(i) (4 pts) If $f, g \in H(\mathbb{T})$ are such that $\widehat{f}(n) = 0$ for $n \leq 0$ and $\widehat{g}(n) = 0$ for $n > 0$, then

$$\int_0^{2\pi} f(x)\overline{g(x)}dx = 0?$$

(ii) (4 pts) There is a function $f \in H(\mathbb{T})$ such that

$$\widehat{f}(n) = \frac{1}{|n|^{0.4}} \quad \text{for} \quad n \neq 0?$$

(iii) (4 pts) There is a function $f \in H(\mathbb{T})$ such that

$$\widehat{f}(n) = \frac{2n+3}{2n-1}?$$

Problem 2 Let $f \in H(\mathbb{T})$ be such that $\widehat{f}(0) = 0$, and let

$$F(x) = \int_0^x f(t)dt.$$

(i) (4 pts) Prove that F is 2π-periodic.

(ii) (4 pts) Find $\widehat{F}(n)$ in terms of $\widehat{f}(n)$ for $n \neq 0$.

(iii) (6 pts) Use Fourier inversion I to show that the Fourier series of F converges to F uniformly on $\mathbb{R}$.

Problem 3 Consider the function $f \in H(\mathbb{T})$ given on $[0, 2\pi]$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 2\pi, \\ 0 & \text{if } x = 2\pi. \end{cases}$$

(i) (8 pts) Find the Fourier series of f.

(ii) (6 pts) Find the sum of the Fourier series on $[0, 2\pi]$. (Do not forget to mention which Fourier inversion result you are using; I, II, or III.)

(iii) (8 pts) Show that the Fourier series of f does not converge uniformly on $[0, 2\pi]$. Does the Fourier series of f converge uniformly on $(0, \pi)$?

Problem 4

(i) (10 pts) Solve the regular Sturm-Liouville problem

$$\begin{cases} f'' + \lambda f = 0 \\ f'(0) = 0, \quad f'(l) = 0 \end{cases}$$

on the interval $[0, l]$.

(ii) (10 pts) Solve the BVP

$$\begin{cases} u_t = k u_{xx} & (0 \leq x \leq l,\ t \geq 0) \\ u_x(0,t) = 0, \quad u_x(l,t) = 0 & (t > 0) \\ u(x,0) = f(x) & (0 \leq x \leq l) \end{cases}$$

where k is a positive constant and $f \in L^2([0, l])$.

Problem 5 Suppose that the temperature at time t at a point on the surface of the earth is given by

$$u(0,t) = f(t) = -\frac{5}{2} e^{-2\pi i(365)t} - \frac{7}{2} e^{-2\pi i t} + 10 - \frac{7}{2} e^{2\pi i t} - \frac{5}{2} e^{2\pi i(365)t}.$$

(Here u is measured in °C and t is measured in years.) Suppose that the diffusivity coefficient of the earth is $k = 365\pi$ m^2/yr.

(i) (14 pts) Solve the BVP

$$\begin{cases} u_t = k u_{xx} & (x \geq 0,\ -\infty < t < \infty) \\ u(0,t) = f(t) & (-\infty < t < \infty) \end{cases}$$

to find $u(x,t)$ for $x > 0$.

(ii) (6 pts) At what depth x do the daily variations in temperature become less than $10 \times e^{-1}$ °C ? At what depth x do the annual variations in temperature become less than $14 \times e^{-1}$ °C ?

Problem 6 (12 pts) Use the Fourier transform to solve the BVP

$$\begin{cases} u_{tt} = c^2 u_{xx} & (-\infty < x < \infty,\ t \geq 0) \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & (-\infty < x < \infty). \end{cases}$$

where c is a positive constant and $f, g \in L^1(\mathbb{R})$.