AMERICAN UNIVERSITY OF BEIRUT
MATHEMATICS 224, FINAL EXAMINATION
FALL SEMESTER, 2004-05

Answer the following questions:

1. Let $f_n(x) = nx/(1+nx)$ for $0 \le x < \infty$.
(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$. (2 points)
(b) Does $f_n \to f$ uniformly on $[0,1]$? Justify. (4 points)
(c) Does $f_n \to f$ uniformly on $[1,\infty[$? Justify. (4 points)

2. Use separation of variables to derive the family of solutions

$$u^{\pm}_{mn}(x,y,z) = \sin(m\pi x)\cos(n\pi y)\exp(\pm\sqrt{m^2+n^2}\,\pi z)$$

of the boundary-value problem

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0,$$
$$u(0,y,z) = u(1,y,z) = u_y(x,0,z) = u_y(x,1,z) = 0.$$

(14 points)

3. (a) Define a regular Sturm-Liouville problem on a closed interval $[a,b]$. (3 points)
(b) Define an eigenvalue and eigenfunction of a regular Sturm-Liouville problem. (3 points)
(c) Find the eigenvalues and the normalized eigenfunctions for the following regular Sturm-Liouville problem on $[1,b]$, $b>1$.

$$(x^2 f')' + \lambda f = 0, \quad f(1) = f(b) = 0.$$

Hint: The general solution of the Cauchy-Euler differential equation $ax^2y'' + bxy' + cy = 0$, $x>0$, is $y = c_1x^{r_1} + c_2x^{r_2}$, where r_1, r_2 are the roots of the characteristic equation $ar^2 + (b-a)r + c = 0$. If $r_1 = r_2 = r$, then the general solution is $y = c_1x^r + c_2x^r\ln x$. (14 points)

4. (a) Define the Beta function and show that (5 points)

$$B(x,y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta\,\cos^{2y-1}\theta\,d\theta.$$

(b) Define the Gamma function and show that (5 points)

$$\Gamma(x) = \int_0^1 [\ln(1/t)]^{x-1}\,dt, \quad x>0.$$

(c) State Stirling's formula. (3 points)

5. (a) State the Fourier integral theorem. (3 points)

(b) Conclude from the Fourier integral theorem the Fourier sine and cosine transforms. (3 points)

(c) Show by use of the sine integral that

$$\frac{2}{\pi}e^{-x} = \int_0^\infty \frac{u\sin(xu)}{1+u^2}\,du, \quad x > 0.$$

Hint: $\int e^{ax}\sin(bx)\,dx = e^{ax}(a\sin(bx) - b\cos(bx))/(a^2+b^2) + c$ and part 5(b). (8 points)

6. (a) State Laplace's method. (3 points)

(b) Show that

$$[\Gamma(x+1)]^{1/x} \sim \frac{x}{e} \quad \text{as } x \to \infty.$$

(5 points)

(c) Show that

$$I(x) = \int_0^1 e^{x\cos t}\,dt \sim e^x\sqrt{\frac{\pi}{2x}}.$$

Hint: Show $I(x) = e^x \int_0^1 e^{-x(1-\cos t)}\,dt$. (5 points)

7. The Chebyshev polynomial T_n is defined by

$$T_n(\cos\theta) = \cos(n\theta), \quad n = 0, 1, \cdots.$$

(a) Show that for $n = 0, 1, \cdots$,

$$T_n(x) = \sum_{k \le n/2} \frac{n!}{(2k)!(n-2k)!} x^{n-2k}(x^2-1)^k.$$

Hint: Use $\cos(n\theta) = \Re(\cos\theta + i\sin\theta)^n$ and the Binomial theorem. (5 points)

(b) Show that $\{T_n\}_0^\infty$ is an orthogonal basis with respect to the function $w(x) = (1-x^2)^{-1/2}$ for $L^2(0,\pi)$. (6 points)

(c) Show that for $n = 0, 1, \cdots$,

$$(1-x^2)T_n'' - xT_n' + n^2 T_n = 0.$$

Hint: $\cos(n\theta)$ is a solution for the differential equation $y^{\cdot\cdot} + n^2 y = 0$, where $y^{\cdot\cdot} = d^2y/d\theta^2$. (5 points)

Good Luck and Happy Eid Al Adha