Math 224 – Final Exam (Fall 05) B. Shayya

Problem 1 For $0 < \epsilon \leq \pi$, let f_ϵ be the function in $H(\mathbb{T})$ given on $[-\pi, \pi)$ by

$$f_\epsilon(x) = \begin{cases} 1 - \frac{|x|}{\epsilon} & \text{if } |x| \leq \epsilon, \\ 0 & \text{if } \epsilon < |x| \leq \pi. \end{cases}$$

(i) (5 pts) Show that

$$\widehat{f}(n) = \begin{cases} \frac{\epsilon}{2\pi} & \text{if } n = 0, \\ \frac{2\sin^2(n\epsilon/2)}{\pi\epsilon n^2} & \text{if } n = \pm 1, \pm 2, \ldots. \end{cases}$$

(ii) (3 pts) Find the Fourier series of f_π (so here, $\epsilon = \pi$).

(iii) (4 pts) Write the series of part (ii) in real form.

(iv) (4 pts) Show that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

Problem 2 Let $f \in C(\mathbb{T}) \cap PS(\mathbb{T})$ and consider the BVP

$$(*) \qquad \begin{cases} u_t(x,t) = u_{xx}(x,t), & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = f(x), & x \in \mathbb{R}. \end{cases}$$

(i) (12 pts) Use Fourier series to find a solution of $(*)$.

(ii) Put $f = f_\pi$, where $f_\pi(x) = 1 - |x|/\pi$ is as defined in Problem 1. Notice that since f_π is real-valued and even, a solution $u(x,t)$ of $(*)$ (restricted to the region $[0,\pi] \times [0,\infty)$ in the xt-plane) now represents the temperature at point x and time t of a one-dimensional body that occupies the interval $[0, \pi]$, has initial temperature $u(x, 0) = f_\pi(x)$, and is insulated along its length and at both ends.

(ii)-(a) (6 pts) Use part (i) and part (ii) of Problem 1 to find the temperature $u(x,t)$ for $t > 0$. (It is something of the form $1/2 + \sum_{k=0}^{\infty} a_k(t) \cos(2k+1)x$.)

(ii)-(b) (4 pts) Find a number $T > 0$ such that $0.4 < u(x,t) < 0.6$ for all x when $t > T$.

(ii)-(c) (2 pts) What point x_0 of the body shows no change in temperature as t increases?

(iii) (4 pts) Write the solution obtained in part (i) as a convolution with the heat kernel

$$h_t(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 t} e^{inx}, \qquad x \in \mathbb{R}.$$

(iv) (4 pts) Use the functions f_ϵ from Problem 1 to give a heuristic argument, based on the physical properties of the diffusion of heat, to explain why the heat kernel $h_t(x)$ is expected to be nonnegative everywhere on $\mathbb{R}$.

Problem 3 Let $f \in \mathcal{S}(\mathbb{R}^2)$. Use the Fourier transform to find a solution for each of the following PDEs.

(i) (8 pts) $u_{xx}(x,y) + 2u_{yy}(x,y) + 3u_x(x,y) - 4u(x,y) = f(x,y), \quad (x,y) \in \mathbb{R}^2.$

(ii) (8 pts) $u_{xxxx}(x,y) - u_{yy}(x,y) + 2u(x,y) = f(x,y), \quad (x,y) \in \mathbb{R}^2.$

Problem 4 True or False:

(i) (4 pts) If $f \in C^2(\mathbb{T})$ and $A \in \mathbb{C}$ are such that $f''(x) = Af(x)$ for all $x \in \mathbb{R}$, then $\operatorname{Im} A = 0$?

(ii) (4 pts) If f is a bounded function in $C(\mathbb{R}^d)$ with $\|f\|_{L^1(\mathbb{R}^d)} < \infty$ and $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$, then $f \in \mathcal{S}(\mathbb{R}^d)$?

(iii) (4 pts) There is a function $f \in H(\mathbb{T})$ whose Fourier series is

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n} ?$$

(iv) (4 pts) There is a bounded function f on $\mathbb{R}^d$ with $\|f\|_{L^1(\mathbb{R}^d)} < \infty$ whose Fourier transform is

$$\widehat{f}(\xi) = \frac{1}{(1+|\xi|)^d} ?$$

Problem 5

(i) (8 pts) State and prove Plancherel's theorem.

(ii) (6 pts) Let $f \in \mathcal{S}(\mathbb{R}^8)$. Prove that

$$\int_{\mathbb{R}^5} \left| \int_{\mathbb{R}^8} e^{-2\pi i (\xi,0)\cdot u} f(u)\, du \right|^2 d\xi = \int_{\mathbb{R}^5} \left| \int_{\mathbb{R}^3} f(x,y)\, dy \right|^2 dx.$$

Problem 6 (6 pts) Suppose $\phi \in C(\mathbb{T})$, $\phi(x) \neq 0$ for all $x \in \mathbb{R}$, and

$$\lim_{n\to+\infty} \int_0^{2\pi} e^{-inx^2} \phi(x^2)\, dx = 0.$$

Prove that

$$\lim_{n\to-\infty} \int_0^{2\pi} e^{-inx^2} \phi(x^2)\, dx = 0.$$