Math 213
Final Examination
First Semester 1995-96

1. (a) State and prove Desargues's theorem.
   (b) Let ABCDEF be a hexagon inscribed in a circle. Assume that AB, CD and EF form a triangle UVW. Let L=AB.DE, M=CD.FA and N=BC.EF. Prove that L, M and N are collinear.
   Hint: Consider the three triads of points LDE, AMF and BCN on the sides of UVW.

2. E, F and G are three points in the plane of a circle C of radius R and center O. Locate a point M on C such that

$$ME^2 + 2MF^2 + 3MG^2$$

is a minimum. Explain how the point M is constructed and prove that it satisfies the required property. Give also the construction of the point on C that maximizes the given sum.

3. Given two circles C_1 and C_2 and a point A. Construct a circle passing through A and orthogonal to C_1 and C_2. Use this circle to construct another circle through A tangent to C_1 and C_2. Explain the constructions and prove that they work. If there are cases where the problem has no solution point them out.

4. Let P and P' be two distinct points inverse with respect to a circle C_1 with P inside C_1. Let C_2 be any circle (of center O_2) passing through P and P'. Let A and B be the points of intersection of C_1 and C_2. The perpendicular P'H from P' to O_2P intersects PA and PB in M and N respectively.
   (a) Prove that P' is the mid point of MN.
   (b) If the circle of diameter MN meets C_1 in E and F, show that PP' = PE = PF.

5. Let $\vec{F}(t) = (\sin t)\vec{i} + (\cos t)\vec{j} + ct\,\vec{k}$ be a curve in space.
   (a) Calculate the vectors $\vec{T}$, $\vec{N}$ and $\vec{B}$ for this curve.
   (b) **Find the curvature κ and the torsion τ.**
   Hint: $d\vec{B}/ds = -\tau\vec{N}$.

6. (a) In the hyperbolic plane prove that through a point outside a line two lines may be drawn parallel to the given line.
   (b) If two lines are parallel to the same line they are parallel to each other. Is this statement true in Hyperbolic Geometry? Justify your answer.
   (c) What property does the sum of the angles of a triangle satisfy in Hyperbolic Geometry? Explain.

7. Consider an inversion in the circle of center (0,-1) and radius $\sqrt{2}$.
   (a) What is the image of the interior of the unit circle under this inversion. Justify your answer.
   (b) Consider the hyperbolic triangle in the unit disk whose "vertices" are (0,-1), (1,0) and (0,1). Find the image of this triangle under the given inversion and sketch it. Then find the hyperbolic area.

8. M is the midpoint of side BC of a triangle ABC and AM makes angles e, f, g with AB, AC and BC respectively. Show that sin e/b = sin(e-f)/a cos g.

9. Explain how the Euler characteristic is used in determining the five Platonic solids.