AMERICAN UNIVERSITY OF BEIRUT

The Department of Mathematics

Math 213

Final Examination

January 30, 2004

1. (15 pts.) If $\alpha, \beta, \gamma, \delta$ are the four angles of a convex quadrilateral, prove that

$$\sin\alpha \sin\beta \sin\gamma \sin\delta \leq \frac{64}{\pi^4}\alpha\beta\gamma\delta.$$

Formulate, without proof, a generalization of this inequality.

2. (25 pts.) Let $\boldsymbol{\alpha} : (-b,b) \to R^3$ be a regular curve in R^3 parametrized by arc length. Suppose that

$$a_1|\boldsymbol{\alpha}(s) - \mathbf{u}_1|^2 + a_2|\boldsymbol{\alpha}(s) - \mathbf{u}_2|^2 + a_3|\boldsymbol{\alpha}(s) - \mathbf{u}_3|^2 = c$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are fixed vectors, and a_1, a_2, a_3, and c, are positive constants.

(a) Prove that the trace of $\boldsymbol{\alpha}$ lies on a sphere.

(b) Show that the curvature κ and the torsion τ of $\boldsymbol{\alpha}$ satisfy

$$\frac{1}{\kappa^2(s)} + \frac{1}{\tau^2(s)}[(\frac{1}{\kappa(s)})']^2 = \text{constant.}$$

(c) If $\boldsymbol{\alpha}(0) = \mathbf{0}$, show that $\kappa(s) \geq |\frac{a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3}{a_1 + a_2 + a_3}|^{-1}$.

3. (20 pts.) Let C and D be two points on the x-axis, and (β) the circle of center β and diameter CD. Let A and B be two points in the upper half-plane, inverse of each other with respect to (β).Let (α) be the semicircle in the upper half-plane which passes through A and B, and whose center a lies on the x axis. Let F be the intersection point of (β) and (α). Take a point E on (β) and draw EH perpendicular to the x-axis.Denote by G the intersection of CF and EH.

(a) Taking C as center of inversion, and CE as radius of inversion, find, with justification, the inverse of (β), and the inverse of (α).

(b) Find, with justification, the inverse point A_1 of A, and the inverse point B_1 of B, under the same inversion.

(c) Find the Euclidean distances GA_1, GB_1, HA_1, HB_1 and prove that A_1B_1 is parallel to the x-axis.

(d) Prove that any point M on (β) has equal hyperbolic distances from A and B.

4. (15 pts.) Let $H^2 = \{(x,y) : y > 0\}$, the upper half-plane, with metric $ds = \frac{\sqrt{(dx)^2+(dy)^2}}{y}$.

(a) Specify, without proof, four kinds of transformations under which ds is invariant.

(b) If ABC is the hyperbolic triangle whose sides are given by the three equations: $y = \sqrt{1-x^2}, x = -1$, and $y = \sqrt{4-(x+1)^2}$, sketch the triangle, and find the lengths of the sides, the measures of the angles, and the area.

5. (25 pts.) Let $\boldsymbol{\alpha} : I \to R^3$ be a regular curve in R^3 parametrized by arc length, and $\boldsymbol{\beta} : I \to R^3$ the curve given by

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + (c-s)\mathbf{t}(s)$$

where, c is a real constant, s is the arc length on $\boldsymbol{\alpha}$ and $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are the tangent, normal, and binormal to $\boldsymbol{\alpha}$. Denote by κ, τ the curvature and torsion of α, and by κ_1, τ_1 the curvature and torsion of $\boldsymbol{\beta}$.

(a) Show that $\kappa_1^2 = \frac{\kappa^2+\tau^2}{(c-s)^2\kappa^2}$.

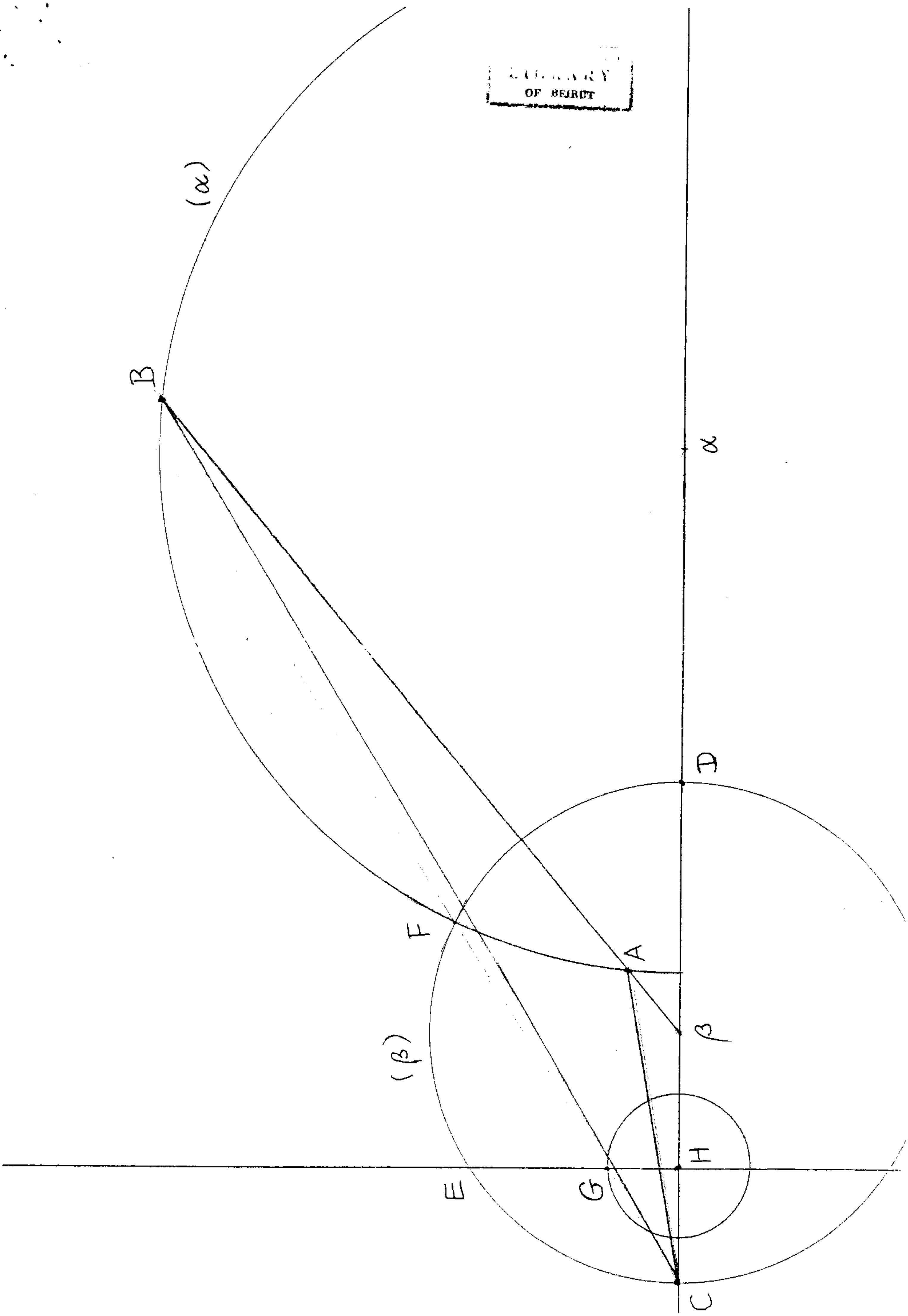
(b) Show that the unit binormal of $\boldsymbol{\beta}$ is $\mathbf{b}_1 = \frac{\kappa\mathbf{b}+\tau\mathbf{t}}{|(c-s)\kappa|\kappa_1}$.

(c) Find the torsion of $\boldsymbol{\beta}$.

6. (Extra credit) Let S be the helicoid surface given by

$$\mathbf{x}(u, v) = (u\cos v, u\sin v, bv), b \neq 0.$$

(a) Show that S is a minimal surface and find its Gaussian curvature.
(b) If L_c is the intersection of S with the plane $z = c$, find the curvature and torsion of L_c.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B
α
D
F
A
β
(β)
H
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G
C