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MATHEMATICS 220
(2nd Semester, 1996-97)
FINAL EXAMINATION

Time: 2 hrs.

June 21, 97

Throughout, V denotes a finite dimensional vector space over a field F.

1. (a) Suppose that A is an orthogonal matrix . Find $|A|$.

   (b) Let B be a 2×2 matrix over $\mathbb{C}$ such that $B^2+2B=3I$. Find $|B|$, and show that B is diagonalizable

   (10 pts.)

2. (a) Let W be a subspace of a vector space V, and suppose that the set of cosets $\{v_1+W, v_2+W, \ldots, v_n+W\}$ is linearly independent in the quotient space V / W. Show that the set of vectors $\{v_1, v_2, \ldots, v_n\}$ in V is linearly independent.

   (b) Give an example to show that the converse of part (a) is false.

   (10 pts.)

3. Let T be a linear operator on $\mathbb{R}^3$ which is represented relative to the standard ordered basis of $\mathbb{R}^3$ by

$$A=\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}$$

   a. Find the characteristic values and the corresponding characteristic spaces for T. Deduce that T is not diagonalizable.
   b. Use the Primary Decomposition Theorem to write $\mathbb{R}^3$ as a direct sum of T-invariant subspaces $\mathbb{R}^3=W_1 \oplus W_2$. Show that W_1 and W_2 are T-cyclic subspaces in this case.
   c. Find the Rational form of A
   d. Find the Jordan form of A.

   (18 pts)

4. Show that all 5×5 complex matrices with characteristic polynomial
   $f(x)= (x-1)^4(x+2)$, and minimal polynomial
   $m(x)= (x-1)^3(x+2)$,
   are similar.

   (9 pts)

5. Let A be an $n\times n$ matrix over $\mathbb{C}$. Prove that if every characteristic value of A is real, then A is similar to a matrix with real entries.

   (9 pts)

6. Let T be a diagonalizable linear operator on V where dim V= n. Prove that
   T has a cyclic vector $\Leftrightarrow$ T has n distinct eigenvalues.

   (10 pts.)

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7. (a) Let T be a linear operator on a real inner product space V such that $T^* = -T$. Show that $\langle T(v), v \rangle = 0$ for all v in V.

(b) Give an example of a linear operator on a real inner product space V such that $\langle T(v), v \rangle = 0$ for every v in V, but $T \neq 0$.

(12 pts.)

8. (a) Prove that characteristic vectors corresponding to distinct characteristic values of a normal operator on a finite dimensional inner product space are orthogonal.

(b) Let V be a complex inner product space, and let T be a linear operator on V. Prove that

T is self-adjoint $\Leftrightarrow$ $\langle T(v), v \rangle$ is real for all v in V.

(12 pts)

9. Answer TRUE or FALSE only (2 points for each correct answer, 0 for no answer, and -1 for each wrong answer).

a. If E is a projection on V, then E is diagonalizable.

b. If T is a linear operator on a finite dimensional inner product space such that $\langle T(u), T(v) \rangle = 0$, for all u,v in V, then T = 0.

c. If U and W are subspaces of a finite dimensional inner product space, then $(U+W)^{\perp} = U^{\perp} \cap W^{\perp}$.

d. Let T be a Hermitian operator on a complex inner product space V. Then T can be represented by a diagonal matix relative to an orthonormal basis of V consisting of characteristic vectors.

e. If A is an n×n matrix over F with minimal polynomial $p(x) = x^3 - x$, then A is invertible.

f. If V is a 2-dimensional vector space over F, and T is a linear operator on V, then the Cyclic Decomposition Theorem gives only one decomposition of V into T-cyclic subspaces.

(12 pts.)