Math 220
Final Exam

June 3, 2004

Throughout V denotes a finite dimensional space over a field K and $T : V \to V$ denotes a linear transformation.

1. (i) Define the minimal polynomial $m(x) = m_T(x) \in K[x]$ of a linear transformation $T: V \to V$.
   (ii) Prove the existence and uniqueness of $m(x)$.
   (iii) If $m(x) = \alpha_0 + \alpha_1 x + \ldots + x^r$, prove that T is an isomorphism if and only if $\alpha_0 \neq 0$.

2. (i) Define what is meant by the statement, an element $\xi \in K$ is a characteristic root of T.
   (ii) Prove that $\xi \in K$ is a characteristic root of T if and only if $m(\xi) = 0$, where $m(x)$ is the minimal polynomial of T.

3. (i) Define what is meant by the statement V is a direct sum of the subspaces V_1 and V_2.
   (ii) Prove that every subspace of V is a direct summand of V.
   (iii) If the linear transformation T is nilpotent, define the index of nilpotency k of T. and prove the existence of $v \in V$ such that the subset $\{v, T(v), \ldots, T^{k-1}(v)\}$ of V in linearly independent.
   (iv) Conclude that $k \leq \dim V$.

4. (i) Let ξ be a characteristic root of T, v the associated characteristic vector and let $u : Kv \to V$ denote the inclusion map. Prove the existence of a linear transformation $p : V \to W$ of vector spaces such that the sequence

$$0 \to Kv \xrightarrow{u} V \xrightarrow{p} W \to 0$$

is exact.

(ii) Let S : $V \to W^*$ be a linear transformation of vector spaces such that Su = 0. Prove the existence of one and only one linear transformation S* : $W \to W^*$ such that S*p = S.

5. State the primary decomposition Theorem associated with the minimal polynomial $m(x)$ of T.

6. (i) If T is nilpotent, prove that the minimal polynomial of T is of the form $m(x) = x^k$, where k is the index of nilpotency of T

   (ii) Make a brief account without proofs why there exists a basis of V such that the matrix of T consists of square matrix blocks each having the entry 1 along the lower super diagonal and 0 entries elsewhere.

7. State the Jordan decomposition theorem of a linear transformation T : $V \to V$.

8. Describe briefly without proofs the Jordan matrix form of a linear transformation.