Math 220

Final Exam

June 3, 2005

Time: 120 minutes.

Throughout T : $V \to V$ is a linear transformation of vector spaces over a field K.

1. (a) Define what is meant by the statement,

   "$V = V_1 \oplus V_2 \oplus \ldots \oplus V_r$ is a ***direct sum*** of the subspaces $V_1, V_2, \ldots, V_r$".

   (b) Let $E_r, \ldots, E_2, E_1 : V \to V$ be linear transformations such that

   (i) $E_r + \ldots + E_2 + E_1 = id_V$, and

   (ii) $E_iE_j = 0$ if $i \neq j$.

   Prove that $V = im\ E_1 \oplus im\ E_2 \oplus \ldots \oplus im\ E_r$

2. If $V = V_1 \oplus V_2$, $B_1 = \{v_1, \ldots, v_r\}$ is a basis of V_1 and $B_2 = \{w_1, \ldots, w_s\}$ a basis of V_2, prove that $B = B_1 \cup B_2$ is a basis of V.

3. (a)Let T : $V \to V$ be a nilpotent linear transformation.. Define what is meant by the statement,

   "A subspace V_1 of a vector space V is T-cyclic of order m".

   (b) If the linear transformation T : $V \to V$ is nilpotent with index of nilpotency equal to m, prove that V has a T-cyclic subspace V_1 of order m. Prove also that $T(V_1)$ is T-cyclic of order $m - 1$.

4. Let T : $V \to V$ be a nilpotent linear transformation of ***index of nilpotency*** n_1.

You already know that V is decomposable as a direct sum $V = V_1 \oplus V_2 \oplus \ldots \oplus V_r$ of subspaces $V_1, V_2, \ldots, V_r$, where every V_i is T-cyclic of order n_i and $n_1 \geq n_2 \geq \ldots \geq n_r$. Prove that the sequence of natural numbers $n_1, \ldots, n_r$ are invariants of T. (That is, prove the following: If $V = U_1 \oplus U_2 \oplus \ldots \oplus U_s$ is a direct sum decomposition of V into T-cyclic subspaces $U_1, \ldots, U_s$ of respective orders $m_1 \geq m_2, \geq \ldots \geq m_s$, then $s = r$ and $n_i = m_i$ for all $i = 1, 2, \ldots, r$).

5. Let $m(x) = (x - \xi_1)^{e_1}(x - \xi_2)^{e_2} \ldots (x - \xi_s)^{e_s}$ be the minimal polynomial of a linear transformation T : $V \to V$.

   (a) State the ***triangular theorem*** associated with $m(x)$.

   (b) State the ***upper triangular theorem*** associated with $m(x)$.

   (c) Describe the ***Jordan blocks*** associated with the factor $(x - \xi_1)^{e_1}$ of $m(x)$ and prove the truth of your description.