Exercise 1: 1st method : (that you should have completely avoided, because it is lengthy and useless!)

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi}(27\cos(10x) - 35\sin(100x))\,dx$$

$$= \frac{27}{2\pi}\underbrace{\int_0^{2\pi}\cos(10x)\,dx}_{=0} - \frac{35}{2\pi}\underbrace{\int_0^{2\pi}\sin(100x)\,dx}_{=0}$$

$$= 0$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi}(27\cos(10x) - 35\sin(100x))\cos(kx)\,dx$$

$$= \frac{27}{\pi}\int_0^{2\pi}\cos(10x)\cos(kx)\,dx - \frac{35}{\pi}\underbrace{\int_0^{2\pi}\sin(100x)\cos(kx)\,dx}_{=0}$$

$$= \begin{cases} 27 & \text{if } k = 10 \\ 0 & \text{if } k \neq 10 \end{cases}$$

since $\displaystyle\int_0^{2\pi}\cos(10x)\cos(kx)\,dx = \begin{cases} \pi & \text{if } k = 10 \\ 0 & \text{if } k \neq 10 \end{cases}$

$$b_k = \frac{1}{\pi}\int_0^{2\pi}(27\cos(10x) - 35\sin(100x))\sin(kx)\,dx$$

$$= \frac{27}{\pi}\underbrace{\int_0^{2\pi}\cos(10x)\sin(kx)\,dx}_{=0} - \frac{35}{\pi}\int_0^{2\pi}\sin(100x)\sin(kx)\,dx$$

$$= \begin{cases} -35 & \text{if } k = 100 \\ 0 & \text{if } k \neq 100 \end{cases}$$

since $\displaystyle\int_0^{2\pi}\sin(100x)\sin(kx)\,dx = \begin{cases} \pi & \text{if } k = 100 \\ 0 & \text{if } k \neq 100 \end{cases}$

The Fourier series $a_0 + \sum_{k\geq 1}(a_k\cos(kx) + b_k\sin(kx))$ generated by f is therefore:

$$a_{10}\cos(10x) + b_{100}\sin(100x)$$
$$= 27\cos(10x) - 35\sin(100x)$$

2nd method : f is already a trigonometric polynomial. Therefore, the Fourier series of f is

$$27 \cos(10x) - 35 \sin(100x).$$

Exercise 2 :

The direction of most rapid increase is given by $\vec{\nabla} f(2,-1)$ so there exists $\lambda \in \mathbb{R}$ such that $\vec{\nabla} f(2,-1) = \lambda \vec{u}$ (in fact $\lambda \geqslant 0$). We deduce that

$$\frac{\partial f}{\partial x}(2,-1)\,\vec{i} + \frac{\partial f}{\partial y}(2,-1)\,\vec{j} = \frac{\lambda}{\sqrt{10}}\,\vec{i} + \frac{3\lambda}{\sqrt{10}}\,\vec{j}$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x}(2,-1) = \dfrac{\lambda}{\sqrt{10}} \\ \dfrac{\partial f}{\partial y}(2,-1) = \dfrac{3\lambda}{\sqrt{10}} \end{cases}$$

. Consequently,

$$6\,\frac{\partial f}{\partial x}(2,-1) - 2\,\frac{\partial f}{\partial y}(2,-1) = \frac{6\lambda}{\sqrt{10}} - \frac{6\lambda}{\sqrt{10}} = 0.$$

Rem : The quantity we have computed is proportional to the directional derivative of f at $(2,-1)$, along the direction of the vector $6\vec{i} - 2\vec{j}$, which is orthogonal to the gradient. Therefore, it is natural to obtain 0.

Exercise 3 :

a) Let $f(\theta) = \frac{1}{1+\cos\theta}$. We have $f(\theta + 2\pi) = f(\theta)$, so f is periodic of period 2π. We may restrict The study to an interval of length 2π.

We take $I =]-\pi, \pi[$.

We have $(r, \theta) \in P \Rightarrow r = \frac{1}{1+\cos\theta}$

$\Rightarrow r = \frac{1}{1+\cos(-\theta)}$

$\Rightarrow (r, -\theta) \in P$

Thus, if we study the curve for $\theta \in [0, \pi[$, the other half of the curve can be deduced by a symmetry with respect to the x-axis.

Finally, $I = [0, \pi[$.

b)

θ	0		$\frac{\pi}{2}$		π
r	$\frac{1}{2}$	$+$	1	$+$	$+\infty$

$r' = \frac{\sin\theta}{(1+\cos\theta)^2}$

. For $\theta = 0$, $r = \frac{1}{2}$ which gives $A(x = \frac{1}{2}, y = 0)$

. For $\theta = \frac{\pi}{2}$, $r = 1$ which gives $B(x = 0, y = 1)$

. Tangent line to P at A : $r' = 0 \Rightarrow \tan V = \frac{\frac{1}{2}}{0} = \infty$

$\Rightarrow$ the tangent line is orthogonal to $\overrightarrow{OA}$.

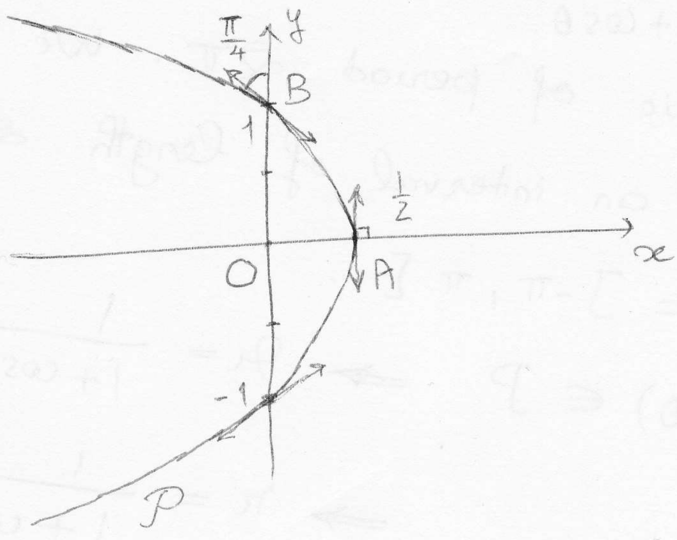
. Tangent line to P at B : $r' = 1 \Rightarrow \tan V = \frac{1}{1} = 1$

$\Rightarrow V = \frac{\pi}{4}$.

Rem : P does not pass by O.

. (c) $\|\overrightarrow{OM}\| = |r| = \dfrac{1}{|1+\cos\theta|} \xrightarrow[\theta \to \pi]{} +\infty$

(d)



$\mathcal{P}$ is a parabola. Indeed,

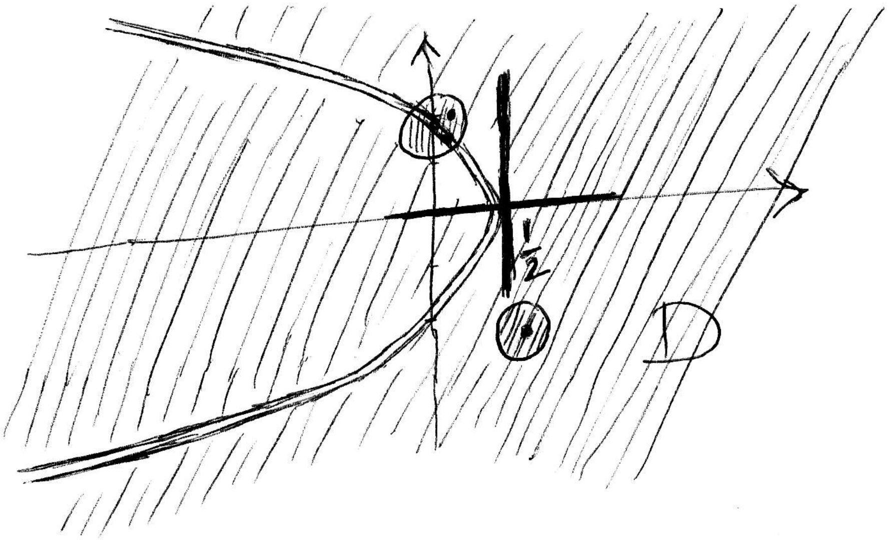
$$r = \frac{1}{1+\cos\theta} \iff r + r\cos\theta = 1$$
$$\iff r = 1 - r\cos\theta$$
$$\implies r^2 = (1 - r\cos\theta)^2$$
$$\iff x^2 + y^2 = (1-x)^2$$
$$\iff \cancel{x^2} + y^2 = 1 - 2x + \cancel{x^2}$$
$$\iff y^2 = 1 - 2x$$

and $y^2 = 1 - 2x$ is a cartesian equation of a parabola.

2. $f(x,y) = \dfrac{\frac{1}{2} - x}{y^2 - 1 + 2x}$

(a) $D = \{ (x,y) \in \mathbb{R}^2 \,/\, y^2 - 1 + 2x \neq 0 \}$

$= \{ (x,y) \in \mathbb{R}^2 \,/\, y^2 \neq 1 - 2x \}$



(b) . D is open because every point of D is an interior point (center of a disk totally contained in D).

. D is not closed because $(0,1)$ is a boundary point of D (any disk centered at $(0,1)$ has points in D (outside P) and outside D (on P)), but $(0,1) \notin D$.

. D is clearly unbounded (no disk can contain D).

(c) . $\displaystyle \lim_{\substack{(x,y) \to (\frac{1}{2},0) \\ y=0}} \frac{\frac{1}{2} - x}{y^2 - 1 + 2x} = \lim_{x \to \frac{1}{2}} \frac{\frac{1}{2} - x}{-1 + 2x}$

$\displaystyle = \lim_{x \to \frac{1}{2}} \frac{\cancel{\frac{1}{2} - x}}{-2(\cancel{\frac{1}{2} - x})} = \lim_{x \to \frac{1}{2}} \left(-\frac{1}{2}\right) = -\frac{1}{2}$

. $\displaystyle \lim_{\substack{(x,y) \to (\frac{1}{2},0) \\ x=\frac{1}{2}}} \frac{\frac{1}{2} - x}{y^2 - 1 + 2x} = \lim_{y \to 0} \frac{0}{y^2} = \lim_{y \to 0} 0 = 0$

Since $-\frac{1}{2} \neq 0$, we deduce that the limit does not exist.

Exercise 4

$$\frac{\partial f}{\partial x}(x,y,z) = 18x$$
$$\frac{\partial f}{\partial y}(x,y,z) = 2y$$
$$\frac{\partial f}{\partial z}(x,y,z) = 8z$$

$$\frac{\partial x}{\partial \theta}(\theta,\varphi) = \frac{1}{3}(\cos\theta)(\cos\varphi)$$
$$\frac{\partial y}{\partial \theta}(\theta,\varphi) = (\cos\theta)(\sin\varphi)$$
$$\frac{\partial z}{\partial \theta}(\theta,\varphi) = -\frac{1}{2}\sin\theta$$

$$\frac{\partial x}{\partial \varphi}(\theta,\varphi) = -\frac{1}{3}(\sin\theta)(\sin\varphi)$$
$$\frac{\partial y}{\partial \varphi}(\theta,\varphi) = (\sin\theta)(\cos\varphi)$$
$$\frac{\partial z}{\partial \varphi}(\theta,\varphi) = 0$$

$$\begin{aligned}
\frac{\partial h}{\partial \theta}(\theta,\varphi) &= \frac{\partial f}{\partial x}(\sigma(\theta,\varphi))\frac{\partial x}{\partial \theta}(\theta,\varphi) + \frac{\partial f}{\partial y}(\sigma(\theta,\varphi))\frac{\partial y}{\partial \theta}(\theta,\varphi) \\
&\quad + \frac{\partial f}{\partial z}(\sigma(\theta,\varphi))\frac{\partial z}{\partial \theta}(\theta,\varphi) \\
&= \left(18 \times \frac{1}{3}(\sin\theta)(\cos\varphi)\right)\left(\frac{1}{3}(\cos\theta)(\cos\varphi)\right) \\
&\quad + \left(2(\sin\theta)(\sin\varphi)\right)\left((\cos\theta)(\sin\varphi)\right) \\
&\quad + \left(8 \times \frac{1}{2}\cos\theta\right)\left(-\frac{1}{2}\sin\theta\right) \\
&= 2(\sin\theta)(\cos\theta)(\cos^2\varphi) + 2(\sin\theta)(\cos\theta)(\sin^2\varphi) \\
&\quad - 2(\sin\theta)(\cos\theta) \\
&= 0
\end{aligned}$$

$$\frac{\partial h}{\partial \varphi}(\theta,\varphi) = \frac{\partial f}{\partial x}(\sigma(\theta,\varphi))\frac{\partial x}{\partial \varphi}(\theta,\varphi) + \frac{\partial f}{\partial y}(\sigma(\theta,\varphi))\frac{\partial y}{\partial \varphi}(\theta,\varphi)$$
$$+ \frac{\partial f}{\partial z}(\sigma(\theta,\varphi))\frac{\partial z}{\partial \varphi}(\theta,\varphi)$$
$$= \left(18 \times \frac{1}{3}(\sin\theta)(\cos\varphi)\right)\left(-\frac{1}{3}(\sin\theta)(\sin\varphi)\right)$$
$$+ \left(2(\sin\theta)(\sin\varphi)\right)\left((\sin\theta)(\cos\varphi)\right)$$
$$+ \left(8 \times \frac{1}{2}\cos\theta\right) \cdot 0$$
$$= -2(\sin^2\theta)(\cos\varphi)(\sin\varphi) + 2(\sin^2\theta)(\sin\varphi)(\cos\varphi)$$
$$= 0$$

2. A cartesian equation of $\mathcal{E}$ is $f(x,y,z) = f(0,-1,0)$

$$\Longleftrightarrow \boxed{9x^2 + y^2 + 4z^2 = 1} \quad (\mathcal{E})$$

3. $9\, x(\theta,\varphi)^2 + y(\theta,\varphi)^2 + 4\, z(\theta,\varphi)^2$

$$= 9 \times \frac{1}{9}(\sin^2\theta)(\cos^2\varphi) + (\sin^2\theta)(\sin^2\varphi) + 4 \times \frac{1}{4}\cos^2\theta$$
$$= 1$$

Therefore, $\sigma(\theta,\varphi) \in \mathcal{E}$.

4. h represents the "restriction" of f to the surface defined parametrically by σ. But question 3. Shows that this surface is contained in the level surface of f passing by $(0,-1,0)$. The variation of f on a level surface being 0, it is natural to obtain $\frac{\partial h}{\partial \theta}(\theta,\varphi) = 0$ and $\frac{\partial h}{\partial \varphi}(\theta,\varphi) = 0$.

In other words, $h(\theta,\varphi) = 9\, x(\theta,\varphi)^2 + y(\theta,\varphi)^2 + 4z(\theta,\varphi)^2$
$= 1$, therefore $\frac{\partial h}{\partial \theta} = \frac{\partial h}{\partial \varphi} = 0$.

5. $\boxed{\vec{\nabla} f(x,y,z) = (18x)\vec{i} + (2y)\vec{j} + (8z)\vec{k}}$

$$\vec{u} = -\frac{\vec{\nabla} f(0,-1,0)}{\|\vec{\nabla} f(0,-1,0)\|} = +\vec{j}$$

since $\|\vec{\nabla} f(0,-1,0)\| = \|-2\vec{j}\| = 2$

6. Let $\begin{cases} P_0 = (0,-1,0) \\ P = (x,y,z) \end{cases}$. A cartesian equation for P

is: $\vec{\nabla} f(P_0) \cdot \overrightarrow{P_0P} = 0$

$\iff (-2\vec{j}) \cdot (x\vec{i} + (y+1)\vec{j} + z\vec{k}) = 0$

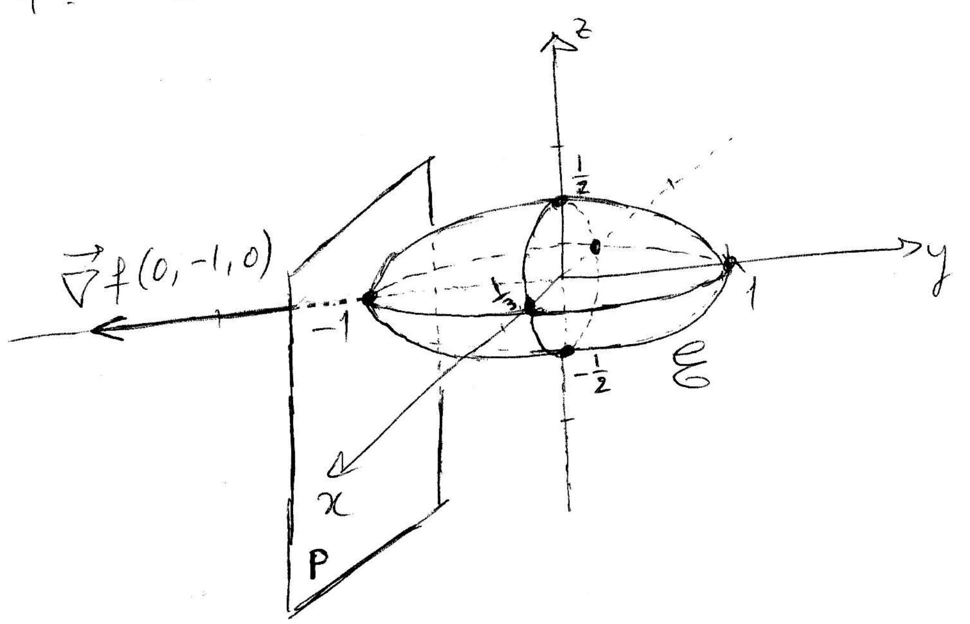
$\iff -2(y+1) = 0$

$\iff \boxed{y = -1}$ (P)

. Parametric equations for the normal line:

$$\boxed{\begin{cases} x = 0 \\ y = -1 - 2t \\ z = 0 \end{cases}}$$

7. $\mathcal{E}$ is an ellipsoid: $\dfrac{x^2}{\left(\frac{1}{3}\right)^2} + \dfrac{y^2}{1^2} + \dfrac{z^2}{\left(\frac{1}{2}\right)^2} = 1$



8. • $(0,-1,0) \in \mathcal{E}$, but $(0,-1,0)$ is not an interior point of $\mathcal{E}$. So $\mathcal{E}$ is not open.

• A point of $\mathbb{R}^3$ is a boundary point of $\mathcal{E}$ if and only if it belongs to $\mathcal{E}$. So $\mathcal{E}$ is closed.

• $\mathcal{E}$ is clearly bounded ($\mathcal{E}$ is contained in the ball $B(0,2)$)

9. The topological boundary of $\mathcal{E}$ is $\mathcal{E}$ itself (cf. question 8).

10. The geometrical boundary of $\mathcal{E}$ does not exist (the ant will never encounter an edge...)