Spring Semester 2013-14
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Prof. H. Abu-Khuzam

MATHEMATICS 242
MIDTERM
Time: 70 Minutes

NAME ------------
ID # ------------

1. Let D be a Euclidean domain and $a \in D$. Prove that

$$d(a) = d(1) \quad \Leftrightarrow \quad a \text{ is a unit in } D$$

[10 points].

Theorem

$d(1) \leq d(1.b) \quad \forall\, b \in D$

($\Leftarrow$) If a is a unit in D

Then $a a^{-1} = 1$

$d(1) = d(a a^{-1})$

$\leq d(a)$

But $d(1)$ is minimal

So $d(a) = d(1)$

($\Rightarrow$) Given: $d(a) = d(1)$

$1 = qa + r$ where $r = 0$ or $d(r) < d(a)$

$d(a) = d(1)$ is minimal $\Rightarrow r = 0$

So $1 = qa$ & so a is a unit

2. Show that the set of all polynomials in Z[x] with constant term 0 is a prime ideal in Z[x].

[10 points].

Given: $P = \{ p(x) \in \mathbb{Z}[x] \mid a_0 = 0 \}$

$p(x) \in P \Rightarrow p(x) = a_n x^n + \cdots + a_1 x = x(a_n x^{n-1} + \cdots + a_1)$

$\in \langle x \rangle$ ideal generated by x

So $P = \langle x \rangle$

x is irreducible in $\mathbb{Z}[x]$

$\Rightarrow \langle x \rangle$ is a maximal ideal in $\mathbb{Z}[x]$

$\Rightarrow \langle x \rangle$ is a prime ideal in $\mathbb{Z}[x]$

3. Prove that in a principal ideal domain (PID), if gcd (a,b) = 1 and a | bc, then a | c.

[10 points].

$\gcd(a,b) = 1 \Rightarrow 1 = ra + sb$ for some $r, s \in D$

$\Rightarrow c = rac + sbc$

$a \mid (ac)$ & $a \mid (bc) \Rightarrow a \mid (rac + sbc) = c$

$\Rightarrow a \mid c$

4. a) Show that $f(x) = 2x^5 + 27x^3 - 18x^2 - 15$ is irreducible in Q[x].

[5 points].

$3 \mid 15$, $3 \mid 18$, $3 \mid 27$, $3 \nmid 2$ X $3^2 \nmid 15$

So $f(x)$ is irreducible by Eisenstein's Theorem.

b) Let $f(x) = x^3 - 2x^2 + x + 1 \in Z_3[x]$. Factor $f(x)$ into irreducible factors in $Z_3[x]$.

[7 points].

$f(0) = 1$, $f(1) = 1-2+1+1 = 1$

$f(2) = 8-8+2+1 = 0 \Rightarrow (x-2)$ is a factor of $f(x)$

$$
\begin{array}{r|l}
 & x^2 + 1 \\
\hline
x-2 & x^3 - 2x^2 + x + 1 \\
 & -\ x^3 \pm 2x^2 \\
\hline
 & x + 1 \\
 & -\ x \pm 2 \\
\hline
 & 3 = 0
\end{array}
$$

So $f(x) = (x-2)(x^2+1)$ irreducible factors in $\mathbb{Z}_3[x]$

5. Let D be an integral domain with field of quotients F. Show that if T is an integral domain such that $D \subset T \subset F$, then F is (isomorphic to the field of quotients of T.

[10 points].

If F is not the smallest field containing T, then there is a field F' such that

$$D \subset T \subset F' \subset F$$

This is a contradiction since F is the field of quotients of D and is the smallest field containing D, while F' contains D and $F' \subset F$.

6. Let G a group of order 99.

a) Show that G has a normal Sylow 3-subgroup H and a normal Sylow 11-subgroup K [6 points].

$|G| = 3^2 \times 11$

* $N^{\underline{o}}$ of Sylow 3-subgps $= 1+3k$, $k = 0, 1, 2, 3$ & divides 11
$= 1$ since k can only be 0

So G has only one sylow 3-subgp, and must be normal

$\Rightarrow H \lhd G$, where H is the sylow 3-subgp, $|H| = 3^2$

* $N^{\underline{o}}$ of Sylow 11-subgps $= 1+11k$, $k = 0, 1, 2, \ldots$ & divides 3^2
$= 1$ (k can only be 0)

$\Rightarrow K \lhd G$, where K is the sylow 11-subgp,
$|K| = 11$

H & K are abelian since their orders are prime (p or p^2)

b) Show that G = HK and deduce that G is abelian. [7 points].

$a \in H \cap K \Rightarrow o(a) \,|\, |H|$ & $o(a) \,|\, |K|$
$\Rightarrow o(a) \,|\, 3^2$ & $o(a) \,|\, 11$
$\Rightarrow o(a) = 1 \Rightarrow a = e$

So $H \cap K = \{e\}$

$|HK| = \dfrac{|H||K|}{|H \cap K|} = |H||K| = 99$

So $G = HK$ & $H \cap K = \{e\}$

So G is the (internal) direct product of H & K

$\Rightarrow$ G is abelian since H & K are abelian

(Note that $\underbrace{hkh^{-1}}_{K}\overbrace{k^{-1}}$ ... $\underbrace{hkh^{-1}k^{-1}}_{H} \in H \cap K \Rightarrow hkh^{-1}k^{-1} = e$
$\Rightarrow hk = kh \quad \forall h \in H, k \in K$)

7. Let D be a principal ideal domain (PID) and a,b are nonzero elements in D. Let
I={ y ∈ D | y is a common multiple of a and b}.
(a) Prove that I is an ideal of D.
(b) Prove that I contains an element m such that m is a least common multiple of a and b. (i.e if a | s and b | s, then m | s)

[12 points].

(a) $I \neq \phi$ since $0 \in I$

$p, q \in I \Rightarrow p = p'a$, $p = p''b$, $q = q'a$, $q = q''b$

$\Rightarrow p - q = p'a - q'a = (p' - q')a$ multiple of a

& $p - q = p''b - q''b = (p'' - q'')b$ multiple of b

So $p - q \in I$

$r \in D, p \in I \Rightarrow rp = rp'a$ multiple of a
$= rp''b$ multiple of b

$\Rightarrow rp \in I$

So I is an ideal

(b) I ideal in PID

$\Rightarrow I = \langle m \rangle$ principal ideal

So m is a common multiple of a & b

If $a | s$ & $b | s \Rightarrow s \in I = \langle m \rangle$

$\Rightarrow m | s$

So m is a least common multiple of a & b

8. Let G be a group of order 6 such that G has only two non-singelton conjugacy classes $C(a_1)$ and $C(a_2)$. Use the class equation to find the order of the center $Z(G)$.

[8 points].

$|G| = |Z(G)| + \sum_{i=1}^{2} [G : N(a_i)]$

$6 = |Z(G)| + |C(a_1)| + |C(a_2)|$

$|C(a_1)| \geq 2$ & $|C(a_2)| \geq 2$ being non singelton classes

So $|Z(G)| = 1$ or $|Z(G)| = 2$

* If $|Z(G)| = 2$, Then $|G/Z(G)| = \frac{6}{2} = 3$ prime

which implies G is abelian (HW Prob)

This is not true since $|G| \neq |Z(G)|$

So $|Z(G)| = 1$

9. Answer TRUE or FALSE <u>only</u> [3 points each] :

a. -F--- The polynomial $f(x) = 2x^3-1$ in $Z_3[x]$ is irreducible over Z_3. $f(0)=2$, $f(1)=1$, $f(2)=$

b. -T--- The ascending chain condition (ACC) holds for ideals in $Z_5[x]$.

c. -T-- If a has a non-singelton conjugacy class, then $[G:N(a)]>1$.

d. -T-- Every proper prime ideal in a PID is maximal.

e -T--- If G is a finite group such that all conjugacy classes in G are singletons, then G is abelian

[15 points].