



Spring Semester 2013-14  
March 28, 2014  
Prof. H. Abu-Khzam

MATHEMATICS 242  
MIDTERM  
Time: 70 Minutes

NAME \_\_\_\_\_  
ID # \_\_\_\_\_

1. Let  $D$  be a Euclidean domain and  $a \in D$ . Prove that  
 $d(a) = d(1) \Leftrightarrow a$  is a unit in  $D$

[ 10 points].

Theorem

$$d(1) \leq d(1, b) \quad \forall b \in D$$

( $\Leftarrow$ ) If  $a$  is a unit in  $D$

$$\text{Then } aa^{-1} = 1$$

$$\begin{aligned} d(1) &= d(aa^{-1}) \\ &\leq d(a) \end{aligned}$$

But  $d(1)$  is minimal

$$\text{So } d(a) = d(1)$$

( $\Rightarrow$ ) Given:  $d(a) = d(1)$

$$1 = qa + r \text{ when } q \neq 0 \text{ or } d(r) < d(a)$$

$$d(a) = d(1) \text{ is minimal} \Rightarrow r = 0$$

So  $1 = qa$  & So  $a$  is a unit



2. Show that the set of all polynomials in  $\mathbb{Z}[x]$  with constant term 0 is a prime ideal in  $\mathbb{Z}[x]$  [10 points].

$$\text{Given: } P = \{ p(x) \in \mathbb{Z}[x] \mid a_0 = 0 \}$$

$p(x) \in P \Rightarrow p(x) = a_n x^n + \dots + a_1 x + a_0 = x(a_n x^{n-1} + \dots + a_1)$   
 $\in \langle x \rangle$  ideal generated by:

$$\text{So } P = \langle x \rangle$$

$x$  is irreducible in  $\mathbb{Z}[x]$

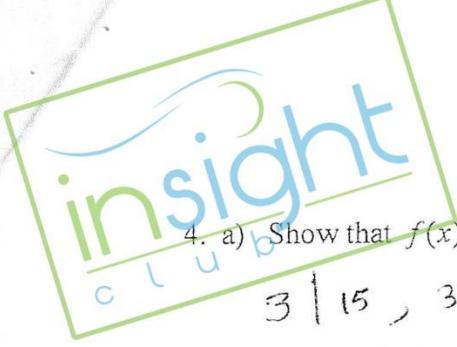
$\Rightarrow \langle x \rangle$  is a maximal ideal in  $\mathbb{Z}[x]$

$\Rightarrow \langle x \rangle$  is a prime ideal in  $\mathbb{Z}[x]$

3. Prove that in a principal ideal domain (PID), if  $\gcd(a, b) = 1$  and  $a \mid bc$ , then  $a \mid c$ . [10 points].

$$\begin{aligned} \gcd(a, b) = 1 &\Rightarrow \exists r, s \in \mathbb{D} \text{ such that } 1 = ra + sb \\ &\Rightarrow c = rac + sbc \end{aligned}$$

$$\begin{aligned} a \mid (ac) \times a \mid (bc) &\Rightarrow a \mid (rac + sbc) = c \\ &\Rightarrow a \mid c \end{aligned}$$



4. a) Show that  $f(x) = 2x^5 + 27x^3 - 18x^2 - 15$  is irreducible in  $\mathbb{Q}[x]$ .

$$3 \nmid 15, 3 \nmid 18, 3 \nmid 27, 3 \nmid 2 \times 3^2 \nmid 15 \quad [5 \text{ points}].$$

so  $f(x)$  is irreducible by Eisenstein's Theorem.

b) Let  $f(x) = x^3 - 2x^2 + x + 1 \in \mathbb{Z}_3[x]$ . Factor  $f(x)$  into irreducible factors in  $\mathbb{Z}_3[x]$ .

[7 points].

$$f(0) = 1, \quad f(1) = 1 - 2 + 1 + 1 = 1$$

$$f(2) = 8 - 8 + 2 + 1 = 1 \Rightarrow (x-2) \text{ is a factor of } f(x)$$

$$\begin{array}{r} x^2 + 1 \\ \hline x-2 \left| \begin{array}{r} x^3 - 2x^2 + x + 1 \\ - x^3 + 2x^2 \\ \hline x + 1 \end{array} \right. \\ \hline x+1 \\ \hline x+2 \\ \hline 3=0 \end{array}$$

$$\text{So } f(x) = (x-2)(x^2+1) \text{ irreducible factors in } \mathbb{Z}_3[x]$$



5. Let  $D$  be an integral domain with field of quotients  $F$ . Show that if  $T$  is an integral domain such that  $D \subset T \subset F$ , then  $F$  is (isomorphic to) the field of quotients of  $T$ .

[10 points].

If  $F$  is not the smallest field containing  $T$ , then there is a field  $F'$  such that

$$D \subset T \subset F' \subset F$$

This is a contradiction since  $F$  is the field of quotients of  $D$  and is the smallest field containing  $D$ , while  $F'$  contains  $D$  and  $F \subset F'$ .

# insight

6. Let  $G$  a group of order 99.

- a) Show that  $G$  has a normal Sylow 3-subgroup  $H$  and a normal Sylow 11-subgroup  $K$  [6 points].

$$|G| = 3^2 \times 11$$

- \*  $N^e$  of Sylow 3-subgps =  $1 + 3k$ ,  $k=0, 1, 2, -$  & divides 11  
 $\leq 1$  since  $k$  can only be 0  
 So  $G$  has only one Sylow 3-subgp, and must be normal  
 $\Rightarrow H \triangleleft G$ , where  $H$  is the Sylow 3-subgp,  $|H| = 3^2$
- \*  $N^e$  of Sylow 11-subgps =  $1 + 11k$ ,  $k=0, 1, 2, -$  & divides  $3^2$   
 $\leq 1$  ( $k$  can only be 0)  
 $\Rightarrow K \triangleleft G$ , where  $K$  is the Sylow 11-subgp,  
 $|K| = 11$   
 $H \times K$  are abelian since their orders are prime ( $p$  or  $p^2$ )

- b) Show that  $G = HK$  and deduce that  $G$  is abelian.

$$a \in HK \Rightarrow o(a) / |H| \times o(a) / |K| \quad [7 \text{ points}].$$

$$\Rightarrow o(a) / 3^2 \times o(a) / 11$$

$$\Rightarrow o(a) = 1 \Rightarrow a = e$$

$$\text{So } HK = \{e\}$$

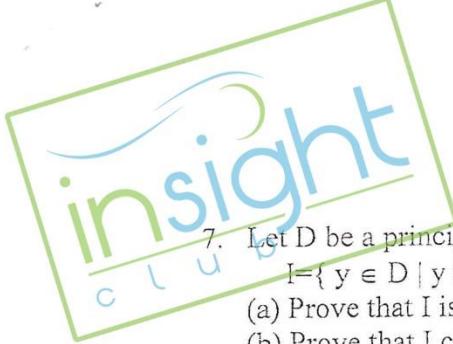
$$|HK| = \frac{|H||K|}{|H \cap K|} = |H||K| = 99$$

$$\text{So } G = HK \quad \times \quad HK = \{e\}$$

So  $G$  is the (internal) direct product of  $H \times K$

$\Rightarrow G$  is abelian since  $H \times K$  are abelian

(Note that  $\overbrace{hkh^{-1}k^{-1}}^H \in HK \Rightarrow hkh^{-1}k^{-1} = e$   
 $\Rightarrow hk = kh \quad \forall h \in H, k \in K$ )



7. Let  $D$  be a principal ideal domain (PID) and  $a, b$  are nonzero elements in  $D$ . Let

$$I = \{y \in D \mid y \text{ is a common multiple of } a \text{ and } b\}.$$

(a) Prove that  $I$  is an ideal of  $D$ .

(b) Prove that  $I$  contains an element  $m$  such that  $m$  is a least common multiple of  $a$  and  $b$ . (i.e if  $a \mid s$  and  $b \mid s$ , then  $m \mid s$ )

[12 points].

(a)  $I \neq \emptyset$  since  $0 \in I$

$$\begin{aligned} p, q \in I &\Rightarrow p = p'a, p = p''b, q = q'a, q = q''b \\ &\Rightarrow p - q = p'a - q'a = (p - q)a \text{ multiple of } a \\ &\quad \& p - q = p''b - q''b = (p'' - q'')b \text{ multiple of } b \end{aligned}$$

$$\text{so } p - q \in I$$

$$r \in D, p \in I \Rightarrow rp = r p'a \text{ multiple of } a \\ = r p''b \text{ multiple of } b$$

$$\Rightarrow r \cdot p \in I$$

so  $I$  is an ideal

(b)  $I$  ideal in PID

$\Rightarrow I = \langle m \rangle$  principal ideal

so  $m$  is a common multiple of  $a \& b$

If  $a \mid s \& b \mid s \Rightarrow s \in I = \langle m \rangle$

$$\Rightarrow m \mid s$$

so  $m$  is a least common multiple of  $a \& b$



8. Let  $G$  be a group of order 6 such that  $G$  has only two non-singleton conjugacy classes  $C(a_1)$  and  $C(a_2)$ . Use the class equation to find the order of the center  $Z(G)$ . [ 8 points].

$$|G| = |Z(G)| + \sum_{i=1}^2 [G : N(a_i)]$$

$$6 = |Z(G)| + |C(a_1)| + |C(a_2)|$$

$|C(a_1)| \geq 2$  &  $|C(a_2)| \geq 2$  being non-singleton classes

$$\text{So } |Z(G)| = 1 \text{ or } |Z(G)| = 2$$

\* If  $|Z(G)| = 2$ , Then  $|G/Z(G)| = \frac{6}{2} = 3$  prime

which implies  $G$  is abelian (HW Prob)

This is not true since  $|G| \neq |Z(G)|$

$$\text{So } |Z(G)| = 1$$

9. Answer TRUE or FALSE only [ 3 points each] :

a. F The polynomial  $f(x) = 2x^3 - 1$  in  $Z_3[x]$  is irreducible over  $Z_3$ .  $\{f(x)\}_{x \in Z_3}$ ,  $\{f(0)=1\}, \{f(1)=2\}$

b. T The ascending chain condition ( ACC ) holds for ideals in  $Z_5[x]$ .

c. F If  $a$  has a non-singleton conjugacy class, then  $[G:N(a)] > 1$ .

d. F Every proper prime ideal in a PID is maximal.

e. T If  $G$  is a finite group such that all conjugacy classes in  $G$  are singletons, then  $G$  is abelian

[ 15 points].