



Spring 2012-13  
Prof. H. Abu-Khuzam  
Time: 75 minutes

MATHEMATICS 242  
Midterm Exam

NAME  
ID

1. Let  $p$  be a prime number. Use the class equation to prove that a group of order  $p^n$ ,  $n \geq 1$ , has a nontrivial center.

[ 12 points].

Class equation:  $|G| = |Z(G)| + \sum_{i=1}^n [G : N(a_i)]$

where,  $a_1, a_2, \dots, a_n$  are the elements of the non-singleton conjugacy class.

$|G| = p^n$ , then  $|G| \text{ mod } p = 0$ .

$|G| = |N(a_i)| [G : N(a_i)] = p^m$

then  $[G : N(a_i)] = p^m$  ( $m < n$ )

then  $[G : N(a_i)] \text{ mod } p = 0$ .

$|Z(G)| = |G| - \sum_{i=1}^n [G : N(a_i)]$

$|G| - \sum_{i=1}^n [G : N(a_i)] \text{ mod } p = 0$

then  $|Z(G)| \text{ mod } p$  should be equal to 0.

$\Rightarrow |Z(G)| \neq 1$

$\Rightarrow G$  has a nontrivial center.



2. Let  $G$  a group of order 45.

a) Show that  $G$  has a normal Sylow 5-subgroup  $H$  and a normal Sylow 3-subgroup  $K$  [10 points].

$$|G| = 45 = 3^2 \times 5$$

\* Number of Sylow 3-subgroups of  $G = 1 + 3k$ ,  $k=0, 1, \dots$  and must divide 5  
 $= 1, 4, 10, \dots$  (only 1 divides 5)

So  $G$  has only one 3-Sylow subgroup  $K$  of order  $3^2 = 9$

$$\text{So } K \triangleleft G \quad \& \quad |K| = 9$$

\* Number of Sylow 5-subgroup of  $G = 1 + 5k$ ,  $k=0, 1, \dots$  and must divide 3  
 $= 1, 6, 11, \dots$  (only 1 divides 3)

So  $G$  has only one Sylow 5-subgroup,  $H$  of order 5

$$\text{So } H \triangleleft G \quad \text{and} \quad |H| = 5$$

b) Show that  $G = HK$  (internal direct product). Deduce that  $G$  is abelian.

[10 points].

\* If  $a \in H \cap K$ , then  $o(a)$  divides  $o(H) = 5$  and  $o(a)$  divides  $o(K) = 3^2$   
But 3 & 5 are relatively prime

$$\text{So } o(a) = 1 \Rightarrow a = e \quad \& \quad H \cap K = \{e\}$$

\*  $G = HK$  :

Clearly  $HK \subseteq G$  &

$$|HK| = \frac{|H||K|}{|H \cap K|} = \frac{5 \times 3^2}{1} = 45$$

$HK$  is a subgroup of  $G$  having the same order as  $G$

$$\text{So } G = HK$$

So  $G = HK$  &  $H \cap K = \{e\}$ , So  $G$  is the internal direct product of  $H$  &  $K$

$|H| = 5 \Rightarrow H$  is cyclic  $\Rightarrow H$  is abelian

$|K| = 3^2 \Rightarrow K$  is abelian

So  $G = HK$  (internal direct product) is abelian



3. Let  $D$  be an integral domain and  $F$  the quotient field of  $D$ . Show that if  $L$  is a field containing  $D$ , then  $L$  contains an isomorphic copy of  $F$ .

[10 points].

$D \subseteq L$ ,  $L$  is a field

$a, b \in D \Rightarrow a, b^{-1} \in L \Rightarrow ab^{-1} \in L$

$\phi: F \rightarrow L$  defined by  $\phi[a/b] = ab^{-1} \in L$   
is a well defined isomorphism of  $F$  onto  $\phi[F]$

$\phi[F]$  is an isomorphic copy of  $F$  inside  $L$

4. Let  $D$  be an integral domain. Prove that if  $p$  is a prime element of  $D$ , then  $p$  is irreducible in  $D[x]$ .

[10 points].

Given:  $p$  is a prime element of  $D$

Suppose  $p$  is reducible in  $D[x]$

$\Rightarrow p = f(x) \cdot g(x)$

$\Rightarrow \deg p = \deg f(x) + \deg g(x)$

$\Rightarrow \deg f(x) + \deg g(x) = 0$

$\Rightarrow \deg f(x) = 0, \deg g(x) = 0$

$\Rightarrow f(x) = a$  (constant)  $g(x) = b$  constant

$\Rightarrow p = f(x) \cdot g(x) = ab$

$p$  divides  $p = ab$ ,  $p$  prime  $\Rightarrow p \mid a$  or  $p \mid b$

$a = pd$

$p = ab$

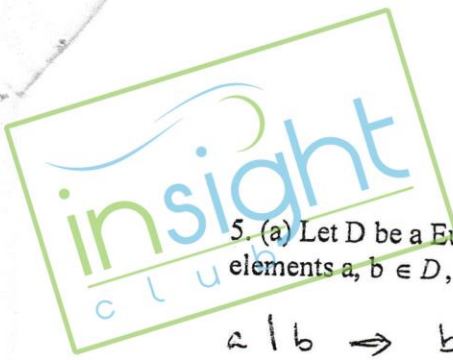
$= pdb$

$\Rightarrow db = 1$

$\Rightarrow b$  unit

or  $b = pd$

$a$  is a unit.



5. (a) Let  $D$  be a Euclidean domain with Euclidean norm  $d$ . Prove that for nonzero elements  $a, b \in D$ , if  $a$  divides  $b$  and  $d(a) = d(b)$ , then  $a$  and  $b$  are associates.

[ 10 points].

(5)

$$a \mid b \Rightarrow b = ac \text{ for some } c \in D$$

$$\Rightarrow d(b) = d(ac)$$

$$\exists q, r \in D \text{ such that } a = bq + r \text{ where } r = 0 \text{ or } d(r) < d(b)$$

$$\Rightarrow a = acq + r \Rightarrow r = a - acq = a(1 - cq)$$

If  $r \neq 0$ , then  $d(r) < d(b) = d(a)$   
 But  $d(r) = d(a(1 - cq)) \geq d(a)$  } contradiction

So  $r = 0$  and  $a = bq$   
 $= acq$

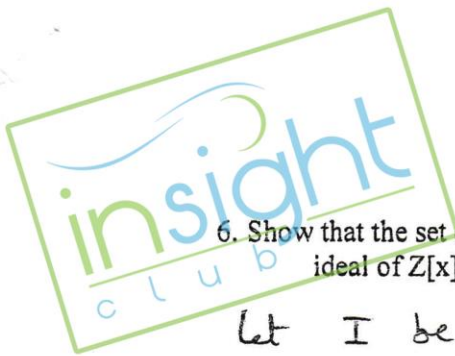
So  $1 = cq \Rightarrow c$  is a unit  
 $\Rightarrow b = ac$ ,  $c$  unit  
 $\Rightarrow a$  &  $b$  are associates

(b) Prove that every Euclidean domain is a PID.

[ 10 points].

(10)

Theorem,



6. Show that the set of all polynomials in  $\mathbb{Z}[x]$  with constant term zero is a prime ideal of  $\mathbb{Z}[x]$ .

Let  $I$  be the ideal of all polynomials in  $\mathbb{Z}[x]$  with constant term zero. [10 points]. (5)

$\Rightarrow I = \{a_n x^n + \dots + a_2 x^2 + a_1 x \mid a_i \in \mathbb{Z}\}$

$= \langle x \rangle$  ideal generated by  $p(x) = x$

$p(x) = x$  is irreducible in  $\mathbb{Z}[x]$

$\Rightarrow \langle x \rangle$  is a maximal ideal

$\Rightarrow I = \langle x \rangle$  is a prime ideal of  $\mathbb{Z}[x]$

Another Method

suppose  $f(x), g(x) \in I$ . Show  $f(x) \in I$  or  $g(x) \in I$

$\Rightarrow$  let  $g(x) = (a_n x^n + \dots + a_1 x + a_0)(b_m x^m + \dots + b_1 x + b_0)$  has constant term

$\Rightarrow$  constant term  $= a_0 b_0 = 0$   $a_0, b_0 \in \mathbb{Z}$

$\Rightarrow a_0 = 0$  or  $b_0 = 0$

$\Rightarrow f(x) \in I$  or  $g(x) \in I$

7. Answer True or False only (3 points for each correct answer and -1 penalty for each wrong answer)

- a. ~~F~~ The polynomial  $f(x) = 2x^3 + x + 2$  is irreducible in  $\mathbb{Z}_5[x]$ .  $f(1) = 5 = 0 \in \mathbb{Z}_5$
- b. ~~T~~ The ascending chain condition (ACC) holds for ideals in  $\mathbb{Z}_5[x]$ .  $\mathbb{Z}_5$  is a field  $\Rightarrow \mathbb{Z}_5[x]$
- c. ~~T~~ A group of order 25 is abelian. order  $p^2$
- d. ~~T~~ If  $G$  is a finite group such that all conjugacy classes in  $G$  are singletons,  $\rightarrow C(a) = \{xax^{-1} \mid x \in G\}$  then  $G$  must be abelian
- e. ~~F~~ A nonabelian group of order 27 has a center of order 9.  $|G/Z| = \frac{27}{9} = 3 \Rightarrow G/Z$  is
- f. ~~T~~  $f(x) = 2x^5 + 27x^3 - 21x^2 - 12$  is irreducible over  $\mathbb{Q}$ . Eisenstein with  $p=3$

[18 points].

(10)