Spring 2012-13	MATHEMATICS 242	NAME
Prof. H. Abu-Khuzam	Midterm Exam	ID
Time: 75 minutes		

1. Let p be a prime number. Use the class equation to prove that a group of order p^n, $n \geq 1$, has a nontrivial center.

[12 points].

Class equation: $|G| = |Z(G)| + \sum_{i=1}^{n} [G : N(a_i)]$

where $a_1, a_2 - - a_n$ are the elements of the non-singleton conjugacy class

$|G| = p^n$, then $|G| \bmod p = 0$

$|G| = |N(a_i)| \, [G : N(a_i)] = p^n$

then $[G : N(a_i)] = p^m$ $(m < n)$

then $[G : N(a_i)] \bmod p = 0$.

$|Z(G)| = |G| - \sum_{i=1}^{n} [G : N(a_i)]$

$|G| - \sum_{i=1}^{n} [G : N(a_i)] \bmod p = 0$

then $|Z(G)| \bmod p$ should be equal to 0.

$\Rightarrow |Z(G)| \neq 1$

$\Rightarrow G$ has a nontrivial center

2. Let G a group of order 45.

a) Show that G has a normal Sylow 5-subgroup H and a normal Sylow 3-subgroup K (10) [10 points].

$|G| = 45 = 3^2 \times 5$

* Number of Sylow 3-subgps of $G = 1+3k$, $k = 0, 1, \ldots$ and must divide 5
$= 1, \cancel{4}, \cancel{7},$ (only 1 divides 5)

So G has only one 3-Sylow subgp K of order $3^2 = 9$
So $K \lhd G$ & $|K| = 9$

* Number of Sylow 5-subgroup of $G = 1+5k$, $k = 0, 1, \ldots$ and must divid
$= 1, \cancel{6}, \cancel{11}$ (only 1 divides 3

So G has only one Sylow 5-subgp, H of order 5
So $H \lhd G$ and $|H| = 5$

b) Show that G = HK (internal direct product). Deduce that G is abelian. (10) [10 points].

* If $a \in H \cap K$, Then $O(a)$ divides $O(H) = 5$
and $O(a)$ divides $O(K) = 3^2$
But 3 & 5 are relatively prime
So $O(a) = 1 \Rightarrow a = e$ & $H \cap K = \{e\}$

* $G = HK$:
Clearly $HK \subseteq G$ &
$$|HK| = \frac{|H||K|}{|H \cap K|} = \frac{5 \times 3^2}{1} = 45$$
HK is a subgp of G having the same order as G
So $G = HK$

So $G = HK$ & $H \cap K = \{e\}$, So G is the internal direct Product of H
$|H| = 5 \Rightarrow H$ is cyclic $\Rightarrow H$ is abelian
$|K| = 3^2 \Rightarrow K$ is abelian
So $G = HK$ (internal direct Product) is abelian

3. Let D be an integral domain and F the quotient field of D. Show that if L is a field containing D, then L contains an isomorphic copy of F.

[10 points].

2

$D \subseteq L$, L is a field

$a, b \in D \Rightarrow a, b^{-1} \in L \Rightarrow ab^{-1} \in L$

$\phi : F \to L$ defined by $\phi[a, b] = ab^{-1} \in L$

is a well defined isomorphism of F onto $\phi[F]$

$\phi[F]$ is an isomorphic copy of F inside L

4. Let D be an integral domain. Prove that if p is a prime element of D, then p is irreducible in D[x].

[10 points].

2

Given: p is a prime element of D

Suppose p is reducible in D [x]

$\Rightarrow p = f(x) . g(x)$

$\Rightarrow \deg . \underset{0}{\underset{\|}{p}} = \deg f(x) + \deg g(x)$

$\Rightarrow \deg f(x) + \deg g(x) = 0$

$\Rightarrow \deg f(x) = 0$, $\deg . g(x) = 0$

$\Rightarrow f(x) = a$ (constant) $g(x) = b$ constant

$\Rightarrow p = f(x) . g(x) = ab$

p divides $p = ab$, p prime $\Rightarrow p \mid a$ or $p \mid b$

$a = pd$	~~of~~ $b = pd$
$p = ab$	
$= pdb$	
$\Rightarrow db = 1$	
$\Rightarrow b$ unit	a is a unit.

5. (a) Let D be a Euclidean domain with Euclidean norm d. Prove that for nonzero elements a, b $\in D$, if a divides b and d(a)= d(b), then a and b are associates.

[10 points].

5

$a \mid b \Rightarrow b = ac$ for some $c \in D$

$\Rightarrow d(b) = d(ac)$

$\exists\, q, r \in D$ such that $a = bq + r$ where $r = 0$ or $d(r) < d(b)$

$\Rightarrow a = acq + r \Rightarrow r = a - acq = a(1-cq)$

If $r \neq 0$, then $d(r) < d(b) = d(a)$

But $d(r) = d(a(1-cq)) \geq d(a)$ } contradiction

So $r = 0$ and $a = bq$

$= acq$

So $1 = cq \Rightarrow c$ is a unit

$\Rightarrow b = ac$, c unit

$\Rightarrow a$ & b are associate

(b) Prove that every Euclidean domain is a PID.

[10 points].

10

Theorem.

6. Show that the set of all polynomials in Z[x] with constant term zero is a prime ideal of Z[x].

[10 points]. (5)

Let I be the ideal of all polynomials in $\mathbb{Z}[x]$ with constant term zero.

$\Rightarrow I = \{a_n x^n + \cdots + a_2 x^2 + a_1 x \mid a_i \in \mathbb{Z}\}$

$= \langle x \rangle$ ideal generated by $p(x) = x$

$p(x) = x$ is irreducible in $\mathbb{Z}[x]$

$\Rightarrow \langle x \rangle$ is a maximal ideal

$\Rightarrow I = \langle x \rangle$ is a prime ideal of $\mathbb{Z}[x]$

Another Method

suppose $f(x) \cdot g(x) \in I$. Show $f(x) \in I$ or $g(x) \in I$

$\Rightarrow f(x) \cdot g(x) = (a_n x^n + \cdots + a_1 x + a_0)(b_m x^m + \cdots + b_1 x + b_0)$ has constant term

$\Rightarrow$ constant term $= a_0 b_0 = 0$ $a_0, b_0 \in \mathbb{Z}$

$\Rightarrow a_0 = 0$ or $b_0 = 0$

$\Rightarrow f(x) \in I$ or $g(x) \in I$

7. Answer **True or False** only (3 points for each correct answer and -1 penality for each wrong answer)

a. --F-- The polynomial $f(x) = 2x^3+x+2$ is irreducible in $Z_5[x]$. $f(1) = 5 = 0$ in $\mathbb{Z}_5$
b. --T-- The ascending chain condition (ACC) holds for ideals in $Z_5[x]$. $\mathbb{Z}_5$ is a field $\Rightarrow \mathbb{Z}_5[x]$
c. --T-- A group of order 25 is abelian. order p^2
d. --T--- If G is a finite group such that all conjugacy classes in G are singletons, then G must be abelian $\Rightarrow C(a) = \{xax^{-1}$ $\Rightarrow xa$
e. --F--- A nonabelian group of order 27 has a center of order 9. $|G/Z| = \frac{27}{9} = 3 \Rightarrow G/Z$ is
f. --T-- $f(x) = 2x^5 + 27x^3 - 21x^2 - 12$ is irreducible over Q. Eisenstein with $p = 3$

[18 points].

(10)