Math 242 — Spring 2003
Final exam, $\sqrt{7} \approx 2.64$ hours

Remarks: Each problem on this exam is worth 10 points, except for the quickies which are worth 5 points each. Please make it clear in your exam booklet which problem you are answering on any given page. The exam is somewhat long; do as much of it as you can, and prioritize your time carefully. **If you cannot solve a given part of a problem, you can still assume the result for any later parts of the same problem.**

Please communicate your ideas clearly and give all necessary justifications for your statements.

Good luck and have a good summer!

QUICKIES (5 points each, only 2 or 3 sentences are required):

Problem Q1. Let $\alpha \in \mathbf{C}$ be a root of $x^5 + 3x - 3$. Show that α is not a constructible number.

Problem Q2. Given a finite group G, define what it means for G to be solvable.

Problem Q3. If K is a finite extension field of $\mathbf{Q}$, briefly explain why there exists α such that $K = \mathbf{Q}(\alpha)$.

Problem Q4. Briefly explain why the field $\mathbf{F}_{2401}$ can be written in the form $\mathbf{F}_7(\alpha)$. (Note that $2401 = 7^4$.)

Problem Q5. Let $R = \mathbf{Z}[i]$ be the ring of Gaussian integers. Why is $R[x]$ a UFD?

Problem Q6. Show that there exists a Galois extension L/K with suitable fields L, K such that $G(L/K)$ is isomorphic to the alternating group A_7. (Suggestion: if you don't see this, first write down an extension with Galois group S_7.)

REGULAR QUESTIONS (10 points each):

Problem R1. Factor $x^{15} + 2x^{10} + 3$ in $\mathbf{F}_5[x]$. (For ease of notation, we'll write $\mathbf{F}_5 = \{0, 1, 2, 3, 4\}$ without putting bars. So 2 really means $\overline{2}$.)

Problem R2. Let F be a field of characteristic p, and let E be a finite extension field of F with $p \nmid [E : F]$. Show that E is separable. (Suggestion: if $\alpha \in E$, let $g(x) = \operatorname{irr}(\alpha, F)$. What can you say about the degree of g and about the derivative g'?)

Problem R3. Let $\alpha \in \overline{\mathbf{F}_5}$ be a root of the irreducible polynomial $f(x) = x^3 + x + 1 \in \mathbf{F}_5[x]$. (You do not have to show that f is irreducible.) Express $\dfrac{1}{\alpha^2 + 3}$ as a polynomial expression in α.

Problem R4. Let E/F be a Galois extension of fields. Assume that $G(E/F)$ is a cyclic group of order 10, with generator σ.

a) Show that there exist exactly two different "proper" intermediate fields K, L. This means that $F \subset K \subset E$ but $K \neq F, E$; similarly for L.

b) Show that each of K and L is Galois over F.

Problem R5. Let $E = \mathbf{Q}(\sqrt[4]{5})$, viewing $E \subset \mathbf{R} \subset \mathbf{C}$.

a) Compute $\{E : \mathbf{Q}\}$, and describe all field embeddings $\psi : E \to \overline{\mathbf{Q}}$ fixing $\mathbf{Q}$.

b) Determine the group $G(E/\mathbf{Q})$.

c) Show that $E_{G(E/\mathbf{Q})} \neq \mathbf{Q}$. Why is this not a contradiction?

Problem R6. We will study the cyclotomic extension $E = \mathbf{Q}(\mu_8)$. Let ζ be a primitive 8th root of 1. Note that $E = \mathbf{Q}(\zeta)$ and that $\mathrm{irr}(\zeta, \mathbf{Q}) = \Phi_4(x) = x^4 + 1$.

a) List all the primitive 8th roots of 1.

b) Determine the structure of $G(E/\mathbf{Q})$.

c) Find all the intermediate fields between E and $\mathbf{Q}$. Express each such intermediate field in the form $\mathbf{Q}(\alpha)$ for a suitable α.

d) Bonus (3pts): show that $\sqrt{2} \in E$.

Problem R7. Let E be the splitting field over $\mathbf{Q}$ of the polynomial $x^5 - 2$. Let ζ be a primitive 5th root of 1, and let $\alpha = \sqrt[5]{2}$; then $E = \mathbf{Q}(\zeta, \alpha)$. We will study the Galois group $G(E/K)$, where $K = \mathbf{Q}(\mu_5) = \mathbf{Q}(\zeta)$.

a) Show that $[E : \mathbf{Q}]$ is a multiple of 20, but that $[E : K] \leq 5$. Conclude that $[E : K] = 5$, and that $x^5 - 2$ is irreducible in $K[x]$ (not just in $\mathbf{Q}[x]$).

b) Show that $G(E/K)$ is cyclic of order 5, and give a generator σ of this group.

c) Bonus (3pts): show that the bigger group $G(E/\mathbf{Q})$ is not commutative.