Math 303 Fall 2002
Final Examination
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1. Let μ be a positive measure on X.
   (a) If f, g are two non-negative measurable functions satisfying
   $$\int_X f d\mu < \infty, \int_X g d\mu < \infty, \text{ and } \int_E f d\mu = \int_E g d\mu$$
   for all measurable subsets of X, prove that $f = g$ a.e.
   (b) If $\{f_n\}$ is a sequence of non-negative measurable functions on X, and $f_n \to f$, and $f_n \leq f$, prove that $\int_X f_n d\mu \to \int_X f d\mu$.
2. Let $f \in L^p(R)$ where $1 \leq p < \infty$.
   (a) Prove that $\lim_{h \to 0} \int_R |f(x+h) - f(x)|^p dx = 0$.
   (b) If f is as in part (a), $g \in L^q(R)$ where q is the index conjugate to p, and
   $$F(x) = \int_R f(x+t) g(t) dt,$$
   prove that F is continuous on R.
3. Find each of the following limits and justify your answer.
   $$\text{(a) } \lim_{n \to \infty} \int_0^\infty \frac{\sin \pi x}{1 + x^n} dx, \text{ (b) } \lim_{n \to \infty} n^2 \int_0^1 (1-x)^n \sin \pi x dx.$$
4. Let f be complex measurable on X and assume that $\mu(X) < \infty$. Let
   $$\| f \|_0 = \int_X \frac{|f|}{1 + |f|} d\mu, \quad \| f \|_{1,\infty} = \sup_{0 < t < \infty} t\mu(\{x \in X : |f(x)| > t\}).$$
   If $t > 0$, prove that
   $$\| f \|_0 \leq \frac{\| f \|_{1,\infty}}{t} + \frac{t}{1+t}\mu(X)$$
   and conclude that $\lim_{n \to \infty} \| f_n \|_0 = 0$ whenever $\lim_{n \to \infty} \| f_n \|_{1,\infty} = 0$.