Math 303 – Final Exam (Fall 03) B. Shayya

Problem 1 (20 pts) Let

$$\Delta = \{f \in L^1(\mathbb{R}) : \int_{-\infty}^{\infty} e^{-x^2} f(x)dx = 1\}.$$

Let g be a nonnegative function in $C_0(\mathbb{R})$ with $\|g\|_{L^1} = \int_{-\infty}^{\infty} g(x)dx = 1$, and for $\epsilon > 0$ define $g_\epsilon(x) = \frac{1}{\epsilon} g(\frac{x}{\epsilon})$.

(i) Prove that Δ is a closed and convex subset of $L^1(\mathbb{R})$.

(ii) Let

$$\Phi(g_\epsilon) = \int_{-\infty}^{\infty} e^{-x^2} g_\epsilon(x)dx.$$

Find $\lim_{\epsilon \to 0} \Phi(g_\epsilon)$.

(iii) Let $\gamma = \inf_{f \in \Delta} \|f\|_{L^1}$. Prove that $\gamma = 1$. (Hint: $\frac{1}{\Phi(g_\epsilon)} g_\epsilon \in \Delta$.)

(iv) Conclude that Δ has no element of least length.

Problem 2 (10 pts) Let

$$\Delta = \{f \in L^2(\mathbb{R}) : \int_{-\infty}^{\infty} e^{-x^2} f(x)dx = 1\}.$$

Then Δ, being a closed and convex subset of $L^2(\mathbb{R})$, has an element h of least length. Find h. (Recall that $\int_{-\infty}^{\infty} e^{-2x^2} dx = \sqrt{\pi/2}$.)

Problem 3 (20 pts) Define $\mu \in M(\mathbb{R}^2)$ by

$$\int f d\mu = \int_0^{\pi} f(\cos\theta, \sin\theta) d\theta \qquad (f \in C(\mathbb{R}^n)).$$

What is

$$\mu(\{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}?$$

Problem 4 (25 pts) Suppose $1 < p < \infty$, $f_n \in L^p(\mathbb{R})$ and $\|f_n\|_{L^p} \leq 1$ for $n = 1, 2, \ldots$, and $f_n(x) \to f(x)$ for a.e. $x \in \mathbb{R}$ as $n \to \infty$. Show that

$$\int_{-\infty}^{\infty} f_n(x) g(x)dx \to \int_{-\infty}^{\infty} f(x)g(x)dx \quad \text{as} \quad n \to \infty$$

for all $g \in L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Problem 5 (25 pts) Let $f \in L^1(\mathbb{R})$ and assume that

$$\lim_{h \to 0} \frac{1}{h} \int_{-\infty}^{\infty} |f(x+h) - f(x)| dx = 0.$$

Show that $f(x) = 0$ for a.e. $x \in \mathbb{R}$.