Math 303–Final Exam (Fall 04)

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Problem 1 (25 pts) Suppose μ is a measure on a set X, $\{f_n\}$ is a sequence of complex-valued measurable functions, and

$$\sup_n |f_n(x)| < \infty \qquad (x \in X).$$

If $\mu(X)$ is finite, prove that to each $\epsilon > 0$ there is a measurable set E and a bound B, $0 < B < \infty$, such that
(i) $\mu(E) < \epsilon$
(ii) $|f_n(x)| \leq B$ for $n \geq 1$ and $x \in E^c$.
In other words, show that off of sets of small measure, the sequence $\{f_n\}$ is uniformly bounded.
What if $\mu(X)$ is infinite ?

Problem 2 (25 pts) Let μ be a measure on a set X. If f is a complex-valued measurable function, put

$$\|f\|_{1,\infty} = \sup_{0<t<\infty} t\mu(\{|f| > t\})$$

and

$$\|f\|_0 = \int \frac{|f|}{1+|f|} d\mu.$$

Let the complex-valued measurable functions $f_1, f_2, \ldots$ be such that

$$\lim_{n\to\infty} \|f_n\|_{1,\infty} = 0.$$

If μ is finite, prove that

$$\lim_{n\to\infty} \|f_n\|_0 = 0.$$

Problem 3 (25 pts) Suppose $1 < p < \infty$ and $f \in L^p(\mathbb{R})$.
(i) Show that, for $x > 0$,

$$\left|\int_0^x f(t)dt\right| \leq \|f\|_{L^p} x^{1-\frac{1}{p}}.$$

(ii) Show that

$$\lim_{x\to\infty} \frac{1}{x^{1-\frac{1}{p}}} \int_0^x f(t)dt = 0.$$

Problem 4 (25 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function. Let E denote the set of all points x in $\mathbb{R}$ at which f is differentiable, i.e.

$$E = \{x \in \mathbb{R} : \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \text{ exists}\}.$$

Prove that E is Lebesgue measurable.