... RY
OF BEIRUT

AMERICAN UNIVERSITY OF BEIRUT

The Department of Mathematics

Math 304

Final Examination, January 26, 2004

1. Consider the annular region $A = A(R_1, R_2) = \{z \in C : R_1 < |z| < R_2\}$ where $0 < R_1 < R_2$. Define ρ on A by $\rho(z) = \frac{1}{(R_2-|z|)(|z|-R_1)}$. Suppose that $f : A \to A$ is holomorphic and $c \in A$ is a fixed point of f. i.e. $f(c) = c$. By formulating and proving, an analogue of the curvature version of Schwarz's lemma for (A, ρ), show that $|f'(c)| \leq 1$.

2. Let $\Omega = \{z = x + iy : x > 0, y > 0\}$, and $U = \{z : |z| < 1\}$. If $f : \Omega \to U$ is holomorphic, how large can $|f'(e^{i\pi/4})|$ be? Find the extremal functions.

3. If $\phi : R \to R$ is continuous and bounded, and $\Omega = \{z = x + iy : x > 0, y > 0\}$ show how to define a function u harmonic in Ω and equal to ϕ on the boundary. i.e. $\lim_{x\to 0+} u(x,y) = \phi(y)$, for each $y > 0$, and $\lim_{y\to 0+} u(x,y) = \phi(x)$, for each $x > 0$. Justify all your assertions.

4. Let $f(z) = \frac{\pi^2}{\sin^2 \pi z} - \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.
(a) Prove that f is an entire function.
(b) Prove that f is bounded on the interval $\{z = x : -\frac{1}{2} \leq x \leq \frac{1}{2}\}$ of the real axis.
(c) Prove that f is bounded on $\{z = iy : -\infty < y < \infty\}$, the y-axis.
(d) Prove that f is bounded on C and that in fact $f = 0$.

5. Prove that

$$e^{az} - e^{bz} = (a-b)e^{\frac{(a+b)z}{2}} z \prod_{n=1}^{\infty} \{1 + \frac{(a-b)^2 z^2}{4n^2\pi^2}\}.$$

6. Let E be a **compact** set in a region Ω of the complex plane, and u a positive harmonic function in Ω. Prove that there exists a constant M, depending only on E and Ω, such that

$$u(z_2) \leq M u(z_1)$$

for any two points $z_1, z_2 \in E$.