Final Examination
Math 304–Fall Semester 2005-06

Answer the following questions:
Every question (a) is allocated 3 points and every question (a) is allocated 7 points

1. (a) State Schwarz's reflection principle.
(b) Show that an entire function that maps the unit circle $|z| = 1$ into the real line is constant.

2. (a) Define the winding number of a closed contour about a given point.
(b) Show that the function $f(z) = (z - a)(z - b)$, where $a, b \in \mathbb{C}$ are distinct, has a single-valued analytic square root but not a single-valued analytic logarithm.

3. (a) Classify the singularities of analytic functions in terms of Laurent series.
(b) Show that a meromorphic function in the extended complex plane must be a rational function.

4. (a) State the argument principle.
(b) Use the argument principle to prove the fundamental theorem of algebra.

5. (a) State Schwarz's lemma.
(b) Show that an analytic function f from the unit disc $\mathbb{D} = \{z : |z| < 1\}$ into the right-half plane $H = \{z : \Re z > 0\}$ satisfies

$$\frac{1 - |z|}{1 + |z|}|f(0)| \leq |f(z)| \leq |f(0)|\frac{1 + |z|}{1 - |z|}, \quad z \in \mathbb{D}$$

and

$$|f'(0)| \leq |2\Re f(0)|.$$

(Hint: Apply Schwarz lemma to the function $(w - f(0))/(w + \overline{f(0)})$).)

6. (a) State the max-min principle for harmonic functions.
(b) Show that if u and v are harmonic functions in an open disc $D(a, R)$ that extend continuously to the closed disc $\overline{D}(a, R)$, such that $u \equiv v$ for all $|z - a| = R$, then u and v are identical.

7. (a) State Mittag Leffler's theorem for the plane.
(b) Show that

$$\frac{\pi}{\cos \pi z} = 2\sum_{n=0}^{\infty}(-1)^n \frac{n+1/2}{(n+1/2)^2 - z^2}.$$

8. (a) State Weierstass's factorization theorem.
(b) Show that

$$\cos \pi z = \prod_{n=0}^{\infty}\left[1 - \frac{z^2}{(n+1/2)^2}\right]$$

and conclude the genus of $\cos \sqrt{\pi z}$.

9. (a) State a necessary and sufficient condition for a family $\mathcal{F}$ of analytic functions of a region Ω to be normal.
(b) Show that a family $\mathcal{F}$ of analytic functions of a region Ω that satisfies

$$\int\int_{\Omega} |f(z)|^2 \, dxdy \leq M,$$

for some constant M, is normal.
(Hint: If $\overline{B}(z_0, R)$ is a closed disc in Ω, then show that

$$|f(z_0)|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|^2 \, d\theta \quad 0 \leq r \leq R,$$

and deduce that $\pi R^2 |f(z_0)|^2 \leq M$.)

10. (a) State the Riemann mapping theorem.
(b) Suppose that $\Omega \neq \mathbb{C}$ is a simply connected domain and f_1 and f_2 are two one-to-one conformal maps from Ω onto the open unit disc $\mathbb{D}$ such that $f_1(z_k) = f_2(z_k)$ at three boundary points z_k of Ω; assuming that both functions extend continuously to $\Omega \bigcup \{z_1, z_2, z_3\}$.