Physics 216 Mathematical Physics
Quiz 2, December 4, 2014, Time: 90 minutes

1. Expand $f(x) = x^2$, $0 < x < 2\pi$, if the period is 2π. Use the result for $f(\pi)$ to prove that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$. (17points)

2. The Fourier series for the function $f(x) = \sin x$, $0 < x < \pi$ is given by

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}$$

   Use this result and Parseval's identity to show that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2(2n+1)^2} = \frac{\pi^2}{16} - \frac{1}{2}$. (14p)

3. Use the Fourier sine integral to solve the equation

$$F_s(\omega) = \int_0^{\infty} f(x) \sin \omega x \, dx = \begin{cases} 1 - \omega, & 0 \le \omega \le 1, \\ 0, & \omega > 1 \end{cases}$$

   to determine the function $f(x)$. (14p).

4. Expand the function $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & -1 < x < 0 \end{cases}$ in a Legendre series $\sum_{l=1}^{\infty} a_l P_l(x)$. Hint: Use the recurrence relation $(2l+1) P_l = P'_{l+1}(x) - P'_{l-1}(x)$, $P_l(1) = 1$ and $P_l(0) = 0$ for l =odd. Express your answer in terms of $P_l(0)$ for l =even. (14p)

5. The generating function for Hermie Polynomials is $g(x,t) = e^{2xt - t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$. Do the integral $\int_{-\infty}^{\infty} e^{-x^2} g^2(x,t) \, dx$ to deduce that $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) \, dx = 0$, $n \neq m$. Use $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. (14p)

6. Consider the Sturm-Liouville equation $\mathcal{L}y = \frac{d}{dx}\left(\alpha(x) \frac{dy(x)}{dx}\right) + \gamma(x) y(x) = 0$ with weight function $w(x) = 1$. Let $y_1(x)$ and $y_2(x)$ be two independent solutions of this equation. Define the Wronskian

$$W(x) = \det \begin{vmatrix} y_1(x) & y_2(x) \\ \frac{dy_1(x)}{dx} & \frac{dy_2(x)}{dx} \end{vmatrix}.$$

   Show that $W(x) = \frac{C}{\alpha(x)}$ where C is constant. Hint: Differentiate $\alpha(x) W(x)$ with respect to x. (15p)

7. Derive the spherical harmonics closure relations

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_l^{m*}(\theta_1, \varphi_1) Y_l^m(\theta_2, \varphi_2) = \delta(\cos\theta_1 - \cos\theta_2) \delta(\varphi_1 - \varphi_2)$$

   (12p)

6) $\mathcal{L}y = \frac{d}{dx}(\alpha y' + \gamma y) + \gamma y \quad y' = \frac{dy}{dx} \quad$ with $(\alpha y')' = -\gamma y$

$w(x) = y_1 y_2' - y_2 y_1'$

$(\alpha w)' = (\alpha(y_1 y_2' - y_2 y_1'))' = (\alpha y_2')' y_1 + (\alpha y_2') y_1' - (\alpha y_1')' y_2 - (\alpha y_1') y_2'$

$= -\gamma y_2 y_1 + \gamma y_1 y_2 = 0$