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EECE 491 – Digital Signal Processing

Problem Set 1 - Solution: Basics of Signals

Problem 1:

$y(t)+6\dot{y}(t)+5\ddot{y}(t)=1$

- Particular Solution: $y_p(t) = 1$ since $y(t)+6\dot{y}(t)+5\ddot{y}(t)=1+6*0+5*0=1$
- Homogeneous solution: $y_h(t) = Ae^{s_1 t} + Be^{s_2 t}$
  - ⇨ s_1 and s_2 are the roots of the second order equation $1 + 6s + 5s^2 = 0$
  - ⇨ $s_1 = -0.2$ and $s_2 = -1$
  - ⇨ $y_h(t) = Ae^{-0.2t} + Be^{-t}$

Thus $y(t) = y_p + y_h = Ae^{-0.2t} + Be^{-t} + 1$

We have two unknowns A and B, and to get the exact values of two unknowns we will need to have 2 equations, thus the need of two initial conditions.

We solve the system of equations generated by the initial conditions:

$$\begin{cases} y(0)=1 \\ y'(0)=1 \end{cases} \rightarrow \begin{cases} A+B=0 \\ -0.2A-B=1 \end{cases} \rightarrow \begin{cases} A=1.25 \\ B=-1.25 \end{cases}$$

And finally we get the solution to be $y(t)=1.25e^{-0.2t}-1.25e^{-t}+1$.

Problem 2:

a) $\sum_{n=0}^{N} \alpha^n = \frac{1-\alpha^{N+1}}{1-\alpha}$

This is a finite sum (N+1 terms), thus it converges for all finite values for α.

b) $\lim_{N\to\infty} \sum_{n=0}^{N} \alpha^n = \lim_{N\to\infty} \frac{1-\alpha^{N+1}}{1-\alpha} = \frac{1}{1-\alpha}$ which converges for $|\alpha|<1$

c) Different approaches can be adapted for this part, such as :

   a. Long division where we divide 1 by $\alpha^2-2\alpha+1$,

   b. Taylor series expansion where you find the n[th] order derivative for $\frac{1}{(1-\alpha)^2}$ and you write the expansion,

   c. Taking the derivative of the formula found in part b,

   d. Expressing $\frac{1}{(1-\alpha)^2}$ as the product $\frac{1}{1-\alpha}$ by itself, and using the formula from part b.

You will get that $\frac{1}{(1-\alpha)^2} = \sum_{k=0}^{\infty}(k+1)\alpha^k$ which converges for $|\alpha|<1$. (Could be easily verified using ratio test)

Problem 3:

a. To get y_1 from x, we first need to shift x(t) two units to the left, and then we shrink it by a factor of 2, thus $y_1(t) = x(2t+2)$
b. To get y_2 from x, we first shift x(t) one unit to the left and then we invert it in time(symmetry with respect to the y-axis), thus $y_2(t) = x(-t+1)$

Problem 4:

One could tackle this problem using different strategies that could eventually show that the proposed new formula for the convolution is not suitable for our systems. One way is to prove that the Commutativity property of the convolution won't hold in case of the new formula. Start by assuming the validity of the new formula. By going to the frequency domain, we could easily show that $f_1 * f_2 \neq f_2 * f_1$.

Proof:

Assume $f(t) = f_1 * f_2 = \int_{-\infty}^{+\infty} f_1(\tau) f_2(\tau - t) d\tau$. We know that F(s) = $F_1(s)$ $F_2(s)$.

But $F(s) = \int_{-\infty}^{+\infty} e^{-st} f(t)dt = \int_{-\infty}^{+\infty} e^{-st} \int_{-\infty}^{+\infty} f_1(\tau) f_2(\tau - t) d\tau dt = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-st} f_1(\tau) f_2(\tau - t) d\tau dt$

Let t' = τ -t ➔ dt' = -dt , So we get that

$$F(s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-s(\tau - t')} f_1(\tau) f_2(t') d\tau dt' = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-s\tau} f_1(\tau) e^{st'} f_2(t') d\tau dt' \neq F_2(s) F_1(s)$$

Problem 5:

a. $F_{Nyquist} = 2*F_{max} = 2 * 360 = 720$ Hz = 720 sample/sec
b. $F_s < F_{Nyquist}$ so we will get aliasing.

$$x[n] = x_a(\frac{n}{F_s}) = \cos(\frac{4\pi}{5}n) + 3\cos(\frac{6\pi}{5}n) = \cos(\frac{4\pi}{5}n) + 3\cos(\frac{4\pi}{5}n) = 4\cos(\frac{4\pi}{5}n)$$

So the frequency of x[n] will be f = 0.4
c. Transmission bitrate = (#samples/sec)*(#bits/sample) = 600 * $\log_2(512)$ = 5400 bits/sec
d. The reconstructed signal will be y = $4\cos(480\pi n)$

Problem 6:

Result after the A/D converter:

$$x[n] = x_a(\frac{n}{F_{s_1}}) = 3\sin(\frac{100\pi}{200}n) + 2\cos(\frac{250\pi}{200}n) = 3\sin(\frac{\pi}{2}n) + 2\cos(\frac{3\pi}{4}n)$$

Result after the D/A Converter:

$$y(t) = 3\sin(\frac{\pi}{2}1000t) + 2\cos(\frac{3\pi}{4}1000t) = 3\sin(500\pi t) + 2\cos(750\pi t)$$

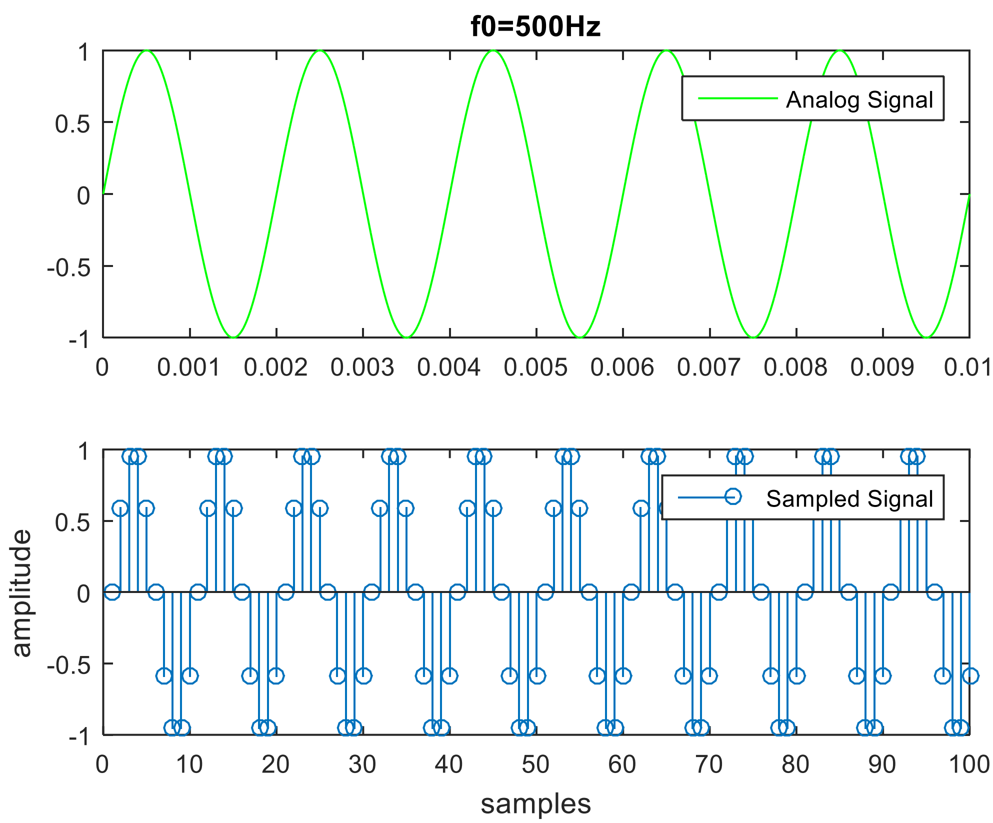
Note that both frequencies present in y are greater than the cutoff frequency of the filter, this they are both removed. We get a zero output.

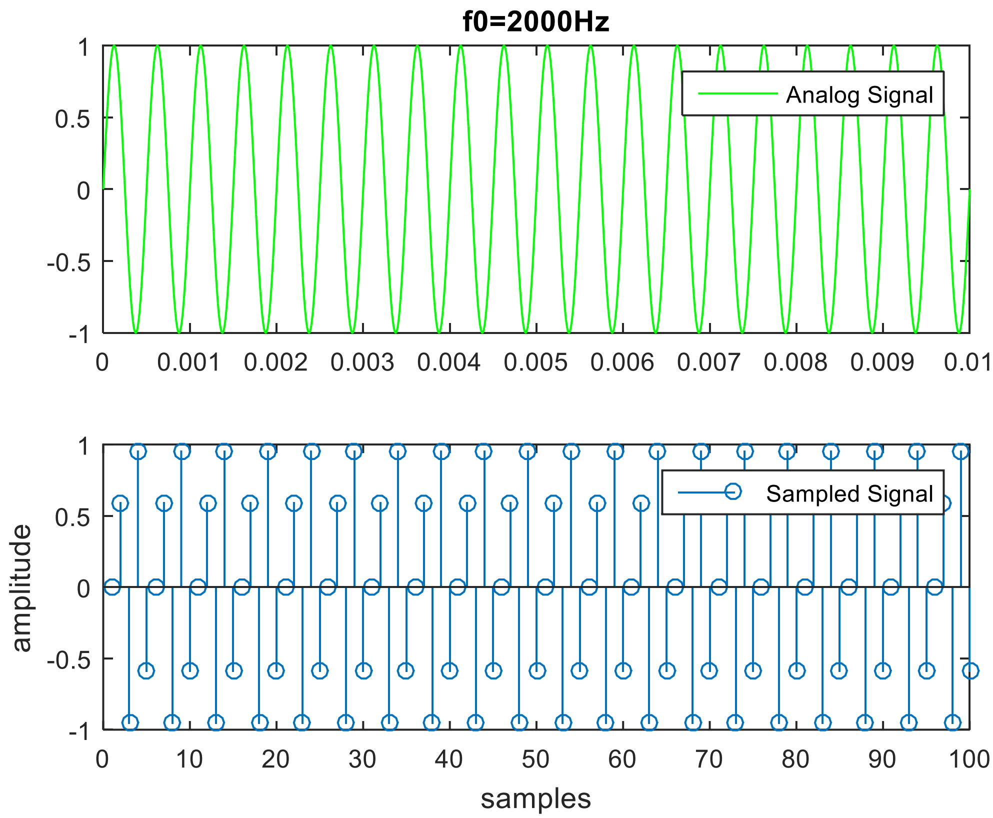
Problem 7:

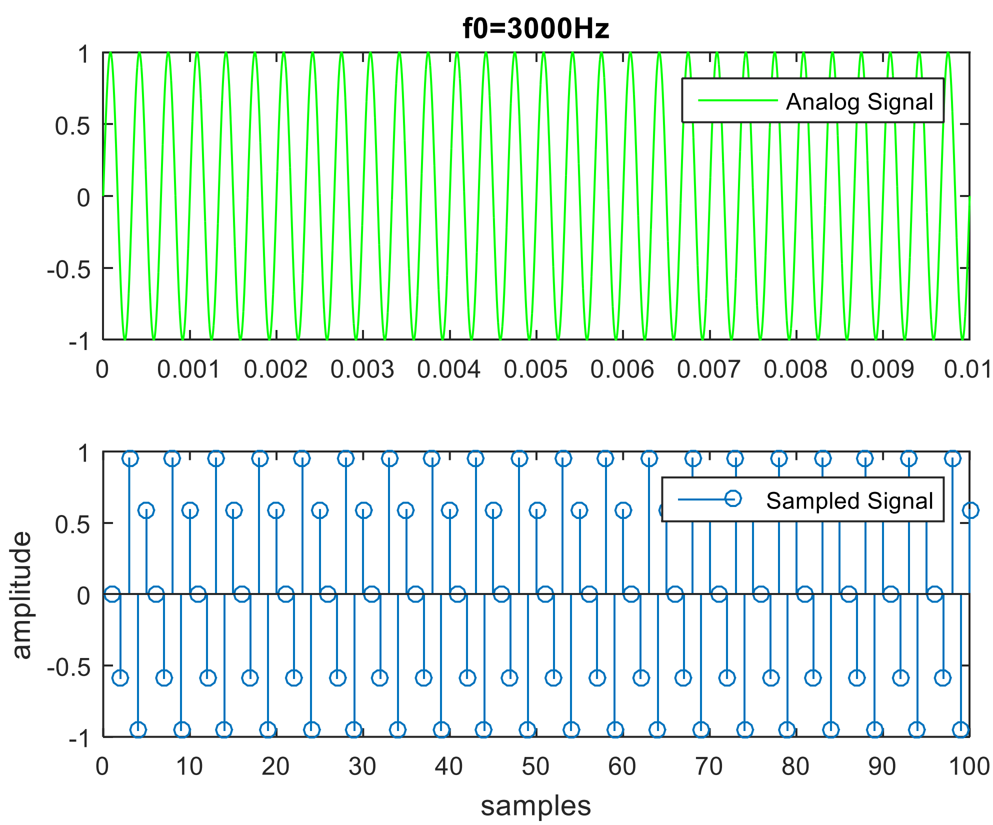
a. Using the code given below, you will get the following figures:

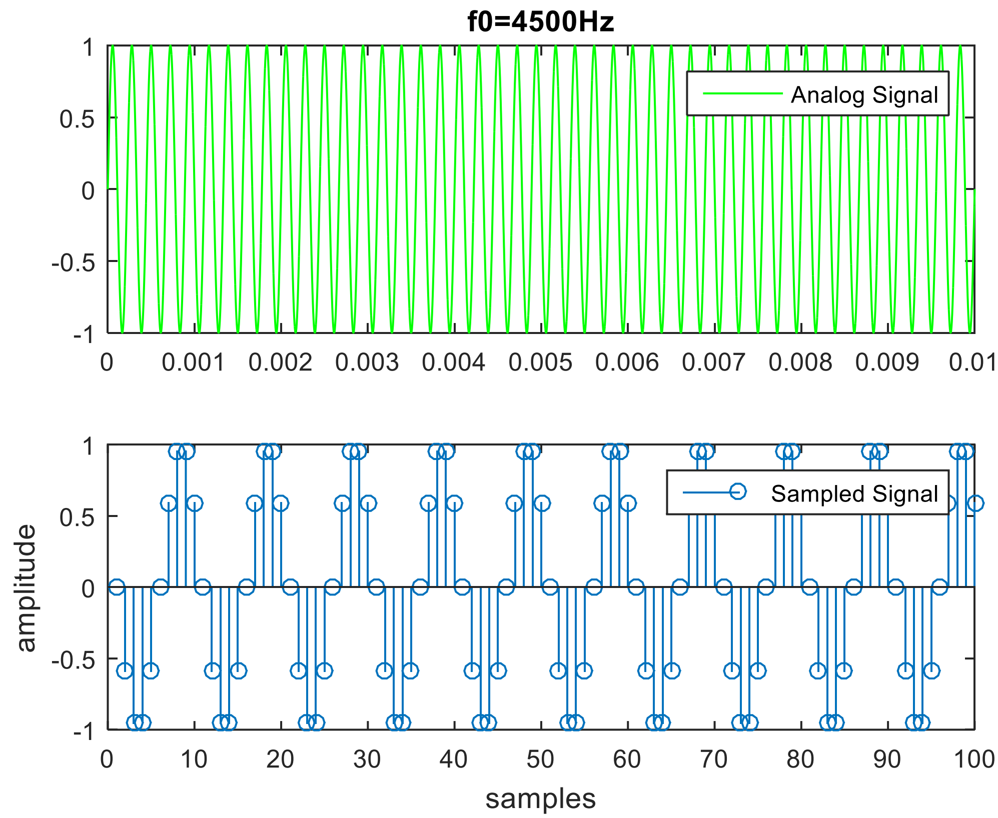
```
f0 = [500 2000 3000 4500];   %% Frequency of Analog input signal
fs = 5000;                   %% Sampling Frequency
t=[0:1e-6:1e-2];             %% Time index
n=[0:1:99];                  %% Sample index


for i=1:length(f0)
figure;
f=sin(2*pi*f0(i)*t);
f1=sin(2*pi*n*f0(i)/fs);
subplot(2,1,1)
plot(t,f,'g');
title(['f0=' num2str(f0(i)) 'Hz']);
legend('Analog Signal');
subplot(2,1,2);
stem(f1);
xlabel('samples');
ylabel('amplitude');
legend('Sampled Signal');
end
```




f0=2000Hz
Analog Signal
Sampled Signal
amplitude
samples


f0=3000Hz
Analog Signal
Sampled Signal
amplitude
samples



b. $x[n] = x_a(\frac{n}{F_{s_1}}) = \sin(\frac{2\pi F_0}{F_s} n) = \sin(\frac{4\pi}{5} n)$ so frequency = 0.4

c. Taking the even samples only is equivalent to sampling the signal with half the sampling rate, thus F'_s = 2.5 KHz.

So $y[n] = x_a(\frac{n}{F'_s}) = \sin(\frac{2\pi * 2F_0}{F_s} n) = \sin(\frac{8\pi}{5} n) = \sin(\frac{-2\pi}{5} n)$, so frequency = 0.2

Using the following code, you can generate the figure below:

```
f0 = 2000;
fs = 5000;

x=sin(2*pi*n*f0/fs);
even = 2*n+2;

figure
stem(y(even(1:50)));
```