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Problem Set 2 - Solution: The z-transform

Problem 1:

a) $\mathrm{H(z)} = \frac{\left(1-\frac{1}{2}z^{-2}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} = -4 + \frac{5-3z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = -4 + \frac{-2}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{7}{\left(1-\frac{1}{4}z^{-1}\right)}$,
$ROC\colon |z| > \frac{1}{2}$

Using the Inverse Z-transform: $h[n] = -4\delta[n] - 2\left(\frac{1}{2}\right)^n u[n] + 7\left(\frac{1}{4}\right)^n u[n]$

b) $\mathrm{H(z)} = \frac{\mathrm{Y(z)}}{\mathrm{X(z)}} = \frac{\left(1-\frac{1}{2}z^{-2}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} = \frac{1-\frac{1}{2}z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$

⇨ $Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) - \frac{1}{2}z^{-2}X(z)$

⇨ $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n] - \frac{1}{2}x[n-2]$

⇨ $y[n] = \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2] + x[n] - \frac{1}{2}x[n-2]$

Problem 2:

$$\mathrm{H(z)} = \frac{3-7z^{-1}+5z^{-2}}{1-\frac{5}{2}z^{-1}+z^{-2}} = 5 + \frac{1}{(1-2z^{-1})} - \frac{3}{\left(1-\frac{1}{2}z^{-1}\right)}$$

⇨ $h[n] = 5\delta[n] - 2^n u[-n-1] - 3 * \left(\frac{1}{2}\right)^n u[n]$

a)

$$y[n] = x[n] * h[n] = u[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

$$= \begin{cases} \sum_{k=-\infty}^{n} 2^k = -2^{n+1} & n < 0 \\ -\sum_{k=-\infty}^{-1} 2^k + 5 - \sum_{k=0}^{n} 3\left(\frac{1}{2}\right)^k = -2 + 3\left(\frac{1}{2}\right)^n & n \geq 0 \end{cases}$$

$$= -2u[n] - 2^{n+1}u[-n-1] + 3 * \left(\frac{1}{2}\right)^n u[n]$$

b)

$$\mathrm{Y(z)} = \frac{1}{1-z^{-1}}H(z) = \frac{-2}{1-z^{-1}} + \frac{2}{(1-2z^{-1})} + \frac{3}{(1-\frac{1}{2}z^{-1})}$$

Using the Inverse Z-Transform:

$$y[n] = -2u[n] - 2^{n+1}u[-n-1] + 3 * \left(\frac{1}{2}\right)^n u[n]$$

Problem 3:

$$H(z) = \frac{z^{-1} + z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}$$

$$X(z) = \frac{2}{1-z^{-1}}$$

$$Y(z) = H(z)X(z) = \frac{2(z^{-1} + z^{-2})}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)(1-z^{-1})}$$

$$y[n] = \sum[\text{residues of } Y(z)z^{n-1} \text{ inside } C1\ (\text{radius} > 1)]$$

$$Y(z)z^{n-1} = \frac{2(z^{-1} + z^{-2})z^{n-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)(1-z^{-1})}$$

$$f(z) = Y(z)z^{n-1} = \frac{2(1 + z^{-1})z^{n-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)(1-z^{-1})}$$

$$f(z) = \frac{2(z^{-1} + z^{-2})}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)(1-z^{-1})}, \text{at } n = 1$$

$$f(z) = \frac{2(z^2 + z)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)(z-1)}$$

$$R_1 = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) = -\frac{18}{5}$$

$$R_2 = \lim_{z\to-\frac{1}{3}}\left(z+\frac{1}{3}\right)f(z) = -\frac{2}{5}$$

$$R_3 = \lim_{z\to 1}(z-1)f(z) = 6$$

$$y[n] = R1 + R2 + R3 = 2$$

Problem 4:

a)

$$X(z) = \frac{z^{10}}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{2}\right)^{10}\left(z+\frac{3}{2}\right)^{2}\left(z+\frac{5}{2}\right)\left(z+\frac{7}{2}\right)}, \qquad ROC: \frac{1}{2} < |z| < \frac{3}{2}$$

Since the system is stable, the ROC should include the unit circle.

b)

$$x[n] = \sum\left[residues\ of\ X(z)z^{n-1}\ inside\ C1\ (radius < \frac{3}{2})\right]$$

At n = -8,

$$f(z) = \frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{2}\right)^{10}\left(z+\frac{3}{2}\right)^{2}\left(z+\frac{5}{2}\right)\left(z+\frac{7}{2}\right)}$$

Only $p_1 = 0.5$ is inside the contour of radius < 3/2.

$$R1 = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) = \frac{\frac{1}{2}}{\left(\frac{1}{2}-\frac{3}{2}\right)^{10}\left(\frac{1}{2}+\frac{3}{2}\right)^{2}\left(\frac{1}{2}+\frac{5}{2}\right)\left(\frac{1}{2}+\frac{7}{2}\right)} = \frac{1}{96}$$

And $x[-8] = \frac{1}{96}$

Problem 5:

a) Using the Z-transform pairs:

$$X(z) = -\frac{1}{3} * \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{4}{3} * \frac{1}{1-2z^{-1}} = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})}, \frac{1}{2} < |z| < 2$$

b) Since X(z) has poles at 0.5 and 2, the poles at 1 and -0.5 are due to H(z). Since H(z) is causal, its ROS is |z|>1. The ROC of Y(z) must contain the intersection of the ROC of X(z) and the ROC of Y(z). Hence the ROC of Y(z) is 1 < |z| <2.

c)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1+z^{-1})(1-\frac{1}{2}z^{-1})}{(1-z^{-1})\left(1+\frac{1}{2}z^{-1}\right)} = 1 + \frac{\frac{2}{3}}{(z-1)} + \frac{\frac{-2}{3}}{\left(z+\frac{1}{2}\right)}$$

$$h[n] = \delta[n] + \frac{2}{3}u[n] - \frac{2}{3}\left(-\frac{1}{2}\right)^{n} u[n]$$

d) Yes the system is unstable since there is a pole on the unit circle.

Problem 6:

a) Since y[n] is stable, the ROC must include the unit circle.
Hence, we find that ROC_Y: **½ <|z| <2**.

b) y[n] is two sided: given that its ROC is a ring on the z-plane.

c) Since x[n] is stable, ROC_X must include the unit circle. Also, it has a zero at infinity, so the ROC includes infinity. Thus, **ROC_X: |z|** > ¾.

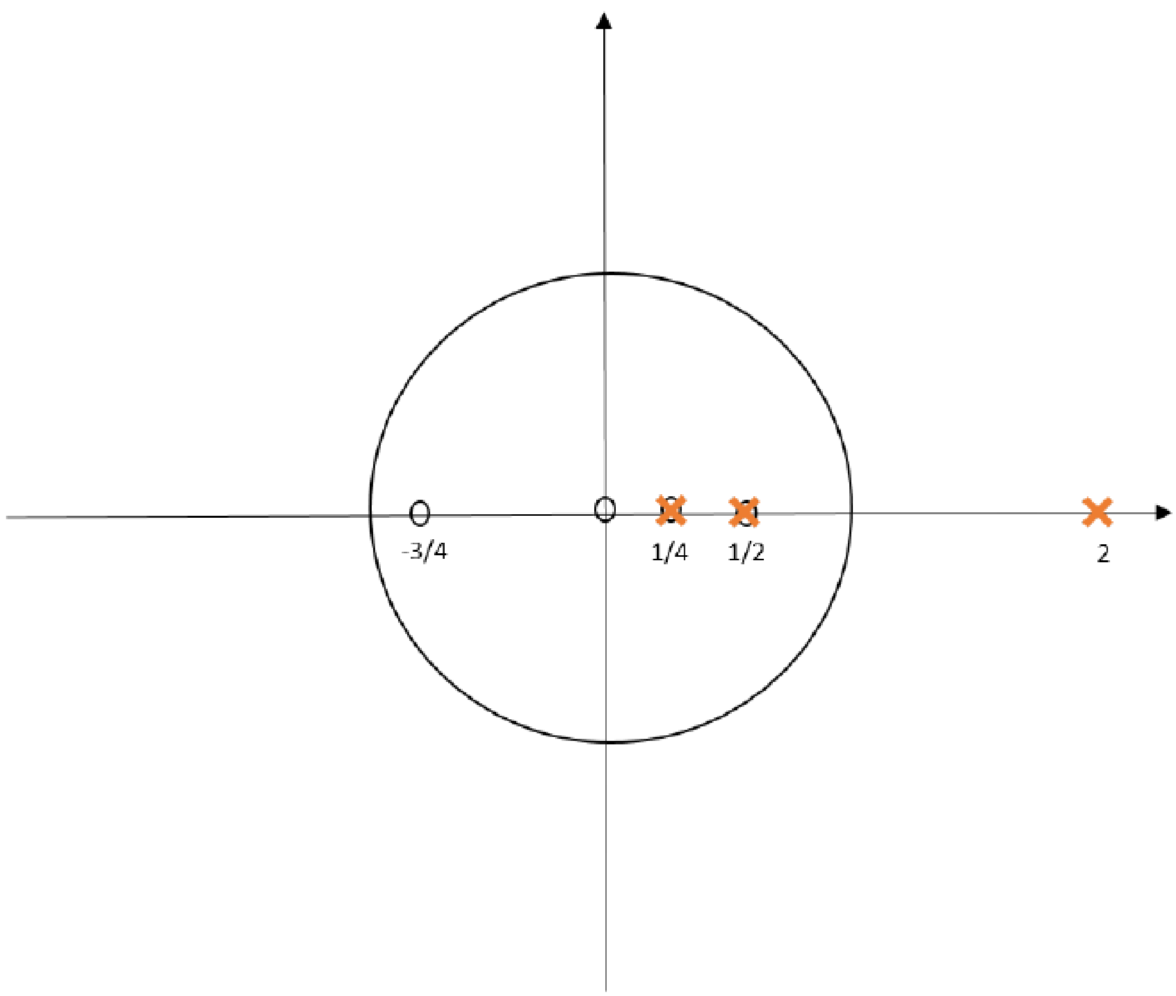
d) Since the ROC of x[n] includes infinity, X(z) contains no positive powers of z, and so x[n] = 0 for n < 0. Therefore x[n] is causal.

e) From the poles/zeros plot we can deduce:

$$\mathrm{X(z)} = \frac{A\left(z-\frac{1}{4}\right)}{\left(z+\frac{3}{4}\right)\left(z-\frac{1}{2}\right)}$$

Using the initial value theorem, x[0] = $X(z)|_{z=\infty}$ = 0

f) ROC_H : |z|< 2



g) ROC_H includes 0 and is therefore anti-causal.