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**Problem Set 3 - Solution**: Frequency Analysis of LTI Systems

**Problem 1:**

a) $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega}$
$= \mathbf{2 + 3e^{-j\omega} + 2e^{-j2\omega}}$

b) $|H(e^{j\omega})|\ (@\ \omega = 0) = |2 + 3e^{-j0} + 2e^{-j2*0}| = 2+3+2 = 7$
$|H(e^{j\omega})|\ \left(@\ \omega = \frac{\pi}{2}\right) = \left|2 + 3e^{-\frac{j\pi}{2}} + 2e^{-j2*\frac{\pi}{2}}\right| = \sqrt{\left(2+(-2)\right)^2 + 3^2} = 3$
$|H(e^{j\omega})|\ (@\ \omega = \pi) = |2 + 3e^{-j\pi} + 2e^{-j2\pi}| = \sqrt{(2+2-3)^2} = 1$

c) If the system is used as a filter, a frequency is considered removed if the magnitude of H at this frequency is less than max(H)/$\sqrt{2}$. The max of H is 7, thus the frequencies attenuated lies in the region $[\frac{\pi}{3}; \pi]$

**Problem 2:**

a)

$$y[n] - \frac{1}{4}y[n-2] = x[n-2] - \frac{1}{4}x[n]$$
$$Y(z) - \frac{1}{4}Y(z)z^{-2} = X(z)z^{-2} - \frac{1}{4}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-2} - \frac{1}{4}}{1 - \frac{1}{4}z^{-2}}, z = e^{j\omega}$$

$$|H(z)|^2 = H(z)H(z^{-1}) = \frac{z^{-2} - \frac{1}{4}}{1 - \frac{1}{4}z^{-2}} * \frac{z^{2} - \frac{1}{4}}{1 - \frac{1}{4}z^{2}} = \frac{1 - \frac{1}{4}(z^2 + z^{-2}) + \frac{1}{16}}{1 - \frac{1}{4}(z^2 + z^{-2}) + \frac{1}{16}} = 1, z = e^{j\omega}$$

$$|H(z)|^2 = 1 \rightarrow |H(z)| = 1$$

The system is therefore an all-pass filter.

b) It is a property of allpass systems that the output energy is equal to the input energy. Here is the proof:

$$\begin{aligned}
\sum_{n=0}^{N-1} |y[n]|^2 &= \sum_{n=-\infty}^{\infty} |y[n]|^2 \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Y(e^{j\omega})\right|^2 d\omega && \text{(by Parseval's Theorem)} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H(e^{j\omega})X(e^{j\omega})\right|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega && \left(\left|H(e^{j\omega})\right|^2 = 1 \text{ since } h[n] \text{ is allpass}\right) \\
&= \sum_{n=-\infty}^{\infty} |x[n]|^2 && \text{(by Parseval's theorem)} \\
&= \sum_{n=0}^{N-1} |x[n]|^2 \\
&= 5
\end{aligned}$$

**Problem 3:**

a)

$$y[n] = x[n] + \alpha x[n-1] + \beta x[n-2] + \gamma[n-3]$$

$$H(z) = \frac{Y(z)}{X(z)} = 1 + \alpha z^{-1} + \beta z^{-2} + \gamma z^{-3}, z = e^{j\omega}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + \alpha e^{-j\omega} + \beta e^{-j2\omega} + \gamma e^{-3j\omega}$$

$$H(e^{j\omega}) = 1 - 0.5e^{-j2\omega}\left(\frac{1}{2}\left(e^{j\omega} + e^{-j\omega}\right)\right) = 1 - 0.25e^{-j\omega} - 0.25e^{-3j\omega}$$

Thus $\alpha = -0.25, \beta = 0, \gamma = -0.25$

b) $W(e^{j\omega}) = Y(e^{j\omega})G(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})G(e^{j\omega})$

We want $\frac{W(e^{j\omega})}{X(e^{j\omega})} = 1 = H(e^{j\omega})G(e^{j\omega})$

Thus $G(e^{-j\omega}) = \frac{1}{H(e^{j\omega})} = \frac{1}{1-0.25e^{-j\omega}-0.25e^{-3j\omega}}$

c) $H(e^{j\omega}) = 1 + \alpha e^{-j\omega} + \beta e^{-j2\omega} + \gamma e^{-3j\omega}$

$$H(e^{j\omega}) = e^{-\frac{3}{2}j\omega}\left(e^{\frac{3}{2}j\omega} + \alpha e^{\frac{1}{2}j\omega} + \beta e^{-\frac{1}{2}j\omega} + \gamma e^{-\frac{3}{2}j\omega}\right)$$

$$H(e^{j\omega}) = 2e^{-\frac{3}{2}j\omega}\left(\frac{e^{\frac{3}{2}j\omega}}{2} + \frac{\alpha e^{\frac{1}{2}j\omega}}{2} + \frac{\beta e^{-\frac{1}{2}j\omega}}{2} + \frac{\gamma e^{-\frac{3}{2}j\omega}}{2}\right)$$

We have 2 cases:

For $\alpha = \beta$ *and* $\gamma = 1,\ H(e^{j\omega}) = 2e^{-\frac{3}{2}j\omega}\left(\cos\left(\frac{3}{2}\omega\right) + \alpha\cos\left(\frac{1}{2}\omega\right)\right)$

For $\alpha = -\beta$ *and* $\gamma = -1, H(e^{j\omega}) = 2e^{-\frac{3}{2}j\omega}\left(\sin\left(\frac{3}{2}\omega\right) + \alpha\sin\left(\frac{1}{2}\omega\right)\right)$

Both of these have linear phase.

**Problem 4:** Note that

$$
\begin{aligned}
& h_2[n] = (-1)^n h_1[n] \\
& \Rightarrow h_2[n] = e^{-j\pi n} h_1[n] \\
& \Rightarrow H_2\left(e^{j\omega}\right) = H_1\left(e^{j(\omega+\pi)}\right)
\end{aligned}
$$

Thus the ideal low pass filter has become an ideal high pass filter.

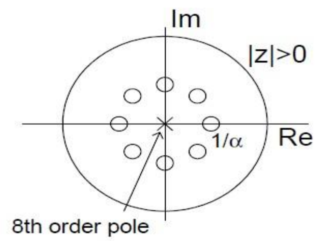
**Problem 5:**

(a)

$$X(z) = S(z)(1 - e^{-8\alpha} z^{-8})$$

$$H_1(z) = 1 - e^{-8\alpha} z^{-8}$$

There are 8 zeros at $z = e^{-\alpha} e^{j\frac{\pi}{4}k}$ for $k = 0, \ldots, 7$
and 8 poles at the origin.




(b)

$$Y(z) = H_2(z)X(z) = H_2(z)H_1(z)S(z)$$

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1}{1 - e^{-8\alpha} z^{-8}}$$

$$|z| > e^{-\alpha} \quad \text{stable and causal,} \qquad |z| < e^{-\alpha} \quad \text{not causal or stable}$$

(c) Only the causal $h_2[n]$ is stable, therefore only it can be used to recover $s[n]$.

$$h[n] = \begin{cases} e^{-\alpha n}, & n = 0, 8, 16, \ldots \\ 0, & \text{otherwise} \end{cases}$$

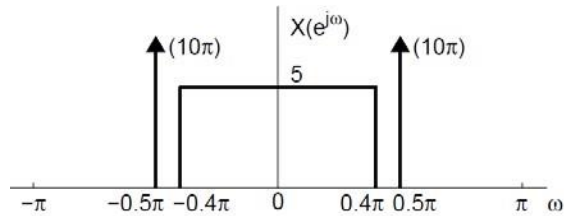
(d)

$$s[n] = \delta[n] \Rightarrow x[n] = \delta[n] - e^{-8\alpha}\delta[n-8]$$

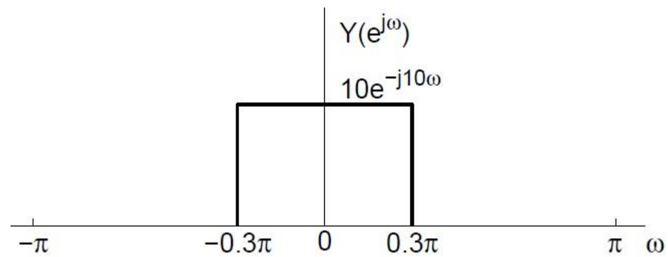
$$\begin{aligned} x[n] * h_2[n] &= \delta[n] - e^{-8\alpha}\delta[n-8] \\ &\quad + e^{-8\alpha}(\delta[n-8] - e^{-8\alpha}\delta[n-16]) \\ &\quad + e^{-16\alpha}(\delta[n-16] - e^{-8\alpha}\delta[n-32]) + \cdots \\ &= \delta[n] \end{aligned}$$

**Problem 6:**

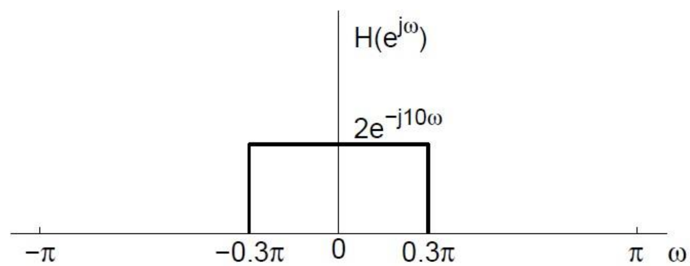
The input $x[n]$ in the frequency domain looks like




while the corresponding output $y[n]$ looks like




Therefore, the filter must be




In the time domain this is

$$y[n] = 2\frac{\sin\left(0.3\pi\left(n-10\right)\right)}{\pi\left(n-10\right)}$$

**Problem 7:**

(a)

| Property | Applies? | Comments |
|---|---|---|
| Stable | No | For a stable, causal system, all poles must be inside the unit circle. |
| IIR | Yes | The system has poles at locations other than $z = 0$ or $z = \infty$. |
| FIR | No | FIR systems can only have poles at $z = 0$ or $z = \infty$. |
| Minimum Phase | No | Minimum phase systems have all poles and zeros located inside the unit circle. |
| Allpass | No | Allpass systems have poles and zeros in conjugate reciprocal pairs. |
| Generalized Linear Phase | No | The causal generalized linear phase systems presented in this chapter are FIR. |
| Positive Group Delay for all $w$ | No | This system is not in the appropriate form. |

(b)

| Property | Applies? | Comments |
|---|---|---|
| Stable | Yes | The ROC for this system function, $\|z\| > 0$, contains the unit circle. (Note there is 7th order pole at $z = 0$). |
| IIR | No | The system has poles only at $z = 0$. |
| FIR | Yes | The system has poles only at $z = 0$. |
| Minimum Phase | No | By definition, a minimum phase system must have all its poles and zeros located *inside* the unit circle. |
| Allpass | No | Note that the zeros on the unit circle will cause the magnitude spectrum to drop zero at certain frequencies. Clearly, this system is not allpass. |
| Generalized Linear Phase | Yes | This is the pole/zero plot of a type II FIR linear phase system. |
| Positive Group Delay for all $w$ | Yes | This system is causal and linear phase. Consequently, its group delay is a positive constant. |

(c)

| Property | Applies? | Comments |
|---|---|---|
| Stable | Yes | All poles are inside the unit circle. Since the system is causal, the ROC includes the unit circle. |
| IIR | Yes | The system has poles at locations other than $z = 0$ or $z = \infty$. |
| FIR | No | FIR systems can only have poles at $z = 0$ or $z = \infty$. |
| Minimum Phase | No | Minimum phase systems have all poles and zeros located inside the unit circle. |
| Allpass | Yes | The poles inside the unit circle have corresponding zeros located at conjugate reciprocal locations. |
| Generalized Linear Phase | No | The causal generalized linear phase systems presented in this chapter are FIR. |
| Positive Group Delay for all $w$ | Yes | Stable allpass systems have positive group delay for all $w$. |