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**EECE 491 – Discrete-time Signal Processing**

**Note on contour integration method for inverse z-transformation**

The $z$-transform is defined by the summation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

and converges in an annular region $R_1 < |z| < R_2$. The inverse $z$ transform is given by

$$x(n) = \frac{1}{2\pi j} \oint_{C_1} X(z) z^{n-1} dz$$

where $C_1$ is any closed path in the region of convergence that encompasses the origin. For example, the path of integration could be a circle of radius $R_1 < C_1 < R_2$.

A theorem from complex variable theory states that a contour integral can be evaluated directly as

$$x(n) = \sum \left[ \text{residues of } X(z) z^{n-1} \text{ inside } C_1 \right]$$

The residue $R_k$ of a complex function $f(z)$ at pole $p_k$ of order 1 is defined as

$$R_k = \lim_{z \to p_k} (z - p_k) f(z)$$

The residue $R_k$ of a complex function $f(z)$ at pole $p_k$ of multiple order $M$ is defined as

$$R_k = \lim_{z \to p_k} \frac{1}{(M-1)!} \frac{d^{M-1}}{dz^{M-1}} \left[ (z - p_k)^M f(z) \right]$$

So, given a rational function $X(z)$ and a specified region of convergence $R_1 < |z| < R_2$, we can compute the inverse $z$-transform $x(n)$ as follows:

1. Make a list of the poles of $z^{n-1}X(z)$ in the region $0 \le |z| < R_1$. Since ROC cannot include poles, and C1 is a circle including origin, the poles must lie in region between origin and $R_1$ (the smaller radius).
2. Compute the residues of these poles and add them up. The result is precisely $x(n)$.

This method is called the *contour integration method* for computing the inverse $z$-transform. It can be somewhat cumbersome because of the poles created at $z = 0$ by the factor $z^{n-1}$ when $n < -1$. These are multiple poles and the multiplicity $n-1$ depends on the time index $n$. Calculating the residues for these can be laborious.

**Example**: Let

$$X(z) = \frac{1}{1 - az^{-1}}$$

For $n \ge 0$, we have a pole of order one at $z = a$. The residue of $z^{n-1} / \left(1 - az^{-1}\right)$ at $z = a$ is

$$R_k = \lim_{z \to a} z^{n-1} \frac{(z-a)}{1 - az^{-1}} = a^n$$

Therefore, $x(n) = a^n$ in this case.

However, if $n<0$, there is a pole of order $n - 1$ at z = 0. We have

$$R_k = \lim_{z \to 0} \frac{1}{(n-2)!} \frac{d^{n-2}}{dz^{n-2}} \left[ \frac{z^{n-1}}{1 - az^{-1}} \right] = 0$$

Therefore, $x(n) = 0$ in this case. Hence,

$$x(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$