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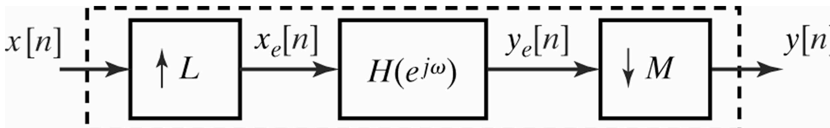
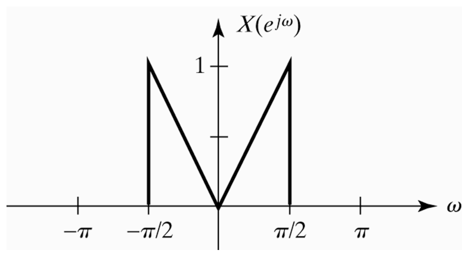
EECE 491 – Discrete-time Signal Processing

Problem Set: Sample Rate Conversion (solutions)

Problem 1: Consider the discrete-time system shown below, where L and M are positive integers. The relationships between $x_e[n]$ and $x[n]$, as well as between $y_e[n]$ and $y[n]$ are as discussed in class. The discrete-time filter $H(e^{j\omega})$ is a low-pass filter with cutoff frequency $\pi/4$ and gain M.

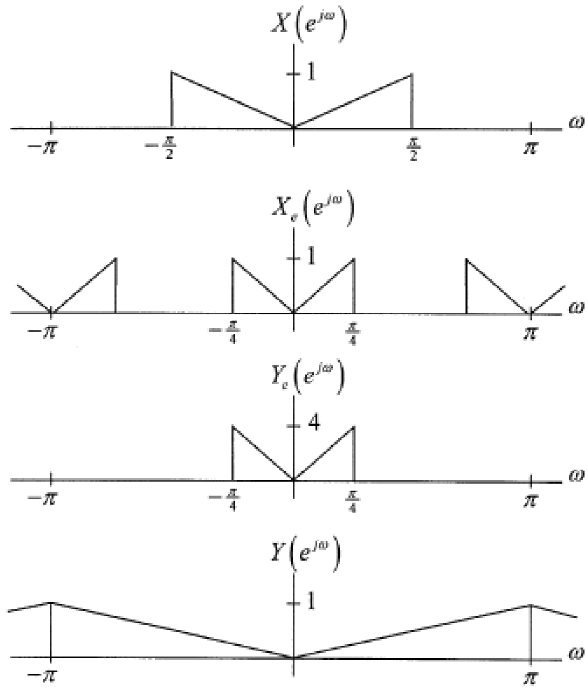


a) Assume the DTFT of the input $X(e^{j\omega})$ is as shown below. Let L=2 and M=4. Sketch $X_e(e^{j\omega}), Y_e(e^{j\omega}), Y(e^{j\omega})$ as a function of ω. Label all your plots.

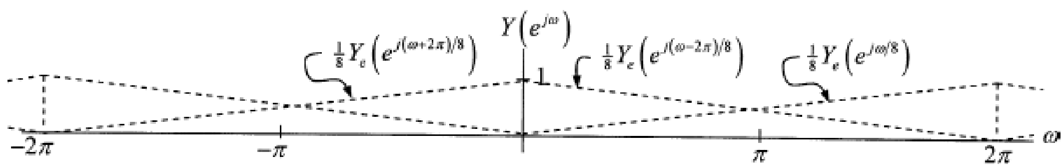


b) Now assume L=2 and M=8. Determine $y[n]$.

Solution: a) With $L = 2$ and $M = 4$, we have

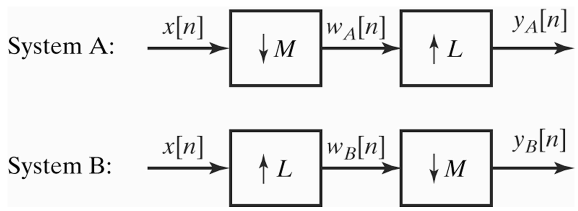


B. With $L = 2$ and $M = 8$, $X_e(e^{j\omega})$ and $Y_e(e^{j\omega})$ remain as in part A, except that $Y_e(e^{j\omega})$ now has a peak value of 8. After expanding we have



We see that $Y(e^{j\omega}) = 1$ for all ω. Inverse transforming gives $y[n] = \delta[n]$ in this case.

Problem 2: Consider the following two discrete-time systems:



a) For M=2 and L=3, and arbitrary input $x[n]$, will $y_A[n] = y_B[n]$?

b) Determine a general condition on M and L to guarantee that $y_A[n] = y_B[n]$ for arbitrary $x[n]$.

Solution:

(a) The following equations describe the stages of System A:

$$w_A[n] = x[2n]$$

$$y_A[n] = \begin{cases} w_A[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

The following equations describe the stages of System B:

$$w_B[n] = \begin{cases} x[\frac{n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

$$y_B[n] = w_B[2n]$$

Therefore,

$$y_A[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$y_B[n] = \begin{cases} x[\frac{2n}{3}] & \text{if } \frac{2n}{3} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Because for all integer values of n for which $\frac{n}{3}$ is an integer, $\frac{2n}{3}$ is also an integer and vice-versa, the systems are equivalent.

(b) More generally, the systems can be described by the following equations:

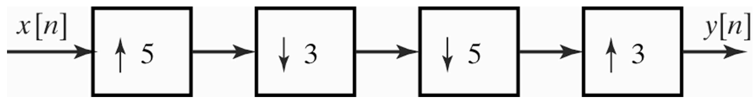
$$y_A[n] = \begin{cases} x[\frac{Mn}{L}] & \text{if } \frac{n}{L} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

$$y_B[n] = \begin{cases} x[\frac{Mn}{L}] & \text{if } \frac{Mn}{L} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

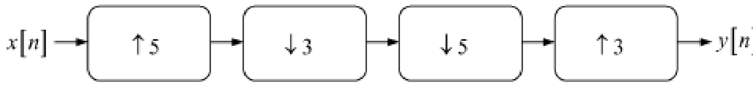
Therefore the two systems are equivalent if for all integer values of n where $\frac{Mn}{L}$ is an integer, $\frac{n}{L}$ is also an integer, and if for all integer values of n where $\frac{n}{L}$ is an integer, $\frac{Mn}{L}$ is also an integer. Since we are guaranteed that for each n which gives integer values of $\frac{n}{L}$, $\frac{Mn}{L}$ must also be an integer (since we're only considering integer M and L), we need only to show that every time $\frac{Mn}{L}$ is an integer, $\frac{n}{L}$ is an integer in order to have an equivalence between the two systems.

For arbitrary integer n, $\frac{Mn}{L}$ is an integer if and only if Mn is an integral multiple of L. This only occurs whenever Mn contains all of L's prime factors. Likewise, $\frac{n}{L}$ is an integer if and only if n contains all of L's prime factors. It is therefore true that in order for the systems to be equivalent, Mn containing all of L's prime factors must imply that n contains all of L's prime factors. This is guaranteed to be true if M and L share no prime factors in common besides 1. (This condition will ensure that any prime factors which Mn has in common with L, besides 1, must have come exclusively from n.) Therefore, the two systems are equivalent if the greatest common factor of M and L is 1 (M and L are co-prime).

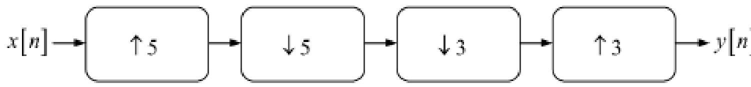
Problem 3: For the system shown below, determine $y[n]$ in terms of $x[n]$. Simplify your answer as much as possible.



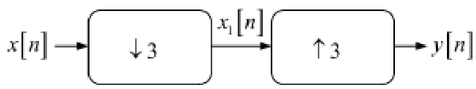
Solution:



Since 3 and 5 are relatively prime, the order of the two operations in the center can be interchanged. This gives



Expanding by 5 and immediately compressing by 5 produces no net effect. We have



Compressing by 3 produces

$$x_1[n] = x[3n].$$

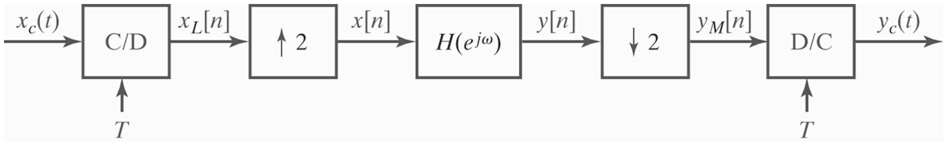
Expanding by 3 now gives

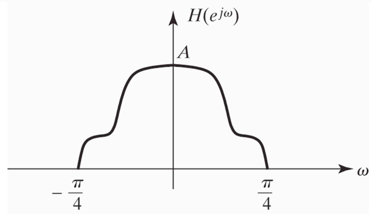
$$y[n] = \begin{cases} x_1[n/3], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$y[n] = \begin{cases} x[n], & n = 3k, \quad k \text{ any integer} \\ 0, & \text{otherwise.} \end{cases}$$

Problem 4: For the system below, assume $X_c(j\Omega)$ is bandlimited to $2\pi(1000)$ and $H(e^{j\omega})$ is as shown below.





a) Determine the most general condition on T so that the overall system from $x_c(t)$ to $y_c(t)$ is LTI. **Hint: Use the Noble identities to simplify your analysis.**

b) Sketch and clearly label the overall effective continuous-time frequency response $H_{eff}(j\Omega)$ when the condition in part a) holds.

c) Now assume that $X_c(j\Omega)$ is bandlimited to avoid aliasing, i.e., $X_c(j\Omega)=0$ for $|\Omega| \geq \pi/T$. For a general sampling period T, we would like to choose the DT system $H(e^{j\omega})$ so that the overall CT system from $x_c(t)$ to $y_c(t)$ is LTI for any input $x_c(t)$ bandlimited as above. Determine the most general condition on $H(e^{j\omega})$ so that the overall system from $x_c(t)$ to $y_c(t)$ is LTI. Assuming that these conditions hold, specify also the overall equivalent CT frequency response $H_{eff}(j\Omega)$ in terms of $H(e^{j\omega})$.

Solution:

a) We can relate $Y_M(e^{j\omega})$ to $X_L(e^{j\omega})$:

$$Y_M(e^{j\omega}) = \frac{1}{2}\left[H(e^{j\frac{\omega}{2}})X_L(e^{j2\frac{\omega}{2}}) + H(e^{j(\frac{\omega}{2}-\pi)})X_L(e^{j2(\frac{\omega}{2}-\pi)})\right]$$

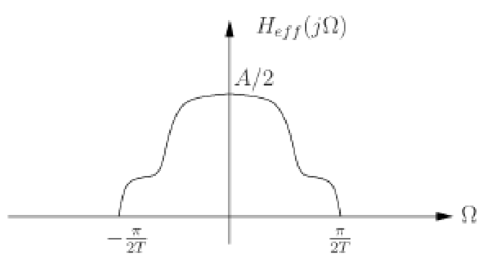
Both $X_L(\cdot)$ terms reduce to $X_L(e^{j\omega})$, the latter because of 2π periodicity, leaving behind the same equivalent frequency response.

The overall CT system will be LTI if there is no aliasing at the C/D converter, or if the DT filter rejects any frequency content contaminated by aliasing. Since the equivalent DT frequency response is 0 for $\frac{\pi}{2} < |\omega| \leq \pi$, the copy of $X_c(j\Omega)$ centred at $\omega = 0$ can extend to $\omega = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$ without passing any aliased components. Applying the $\omega = \Omega T$ mapping:

$$\omega_{max} = \Omega_{max}T = (2\pi \times 10^3)T < \frac{3\pi}{2}$$

and so we require that $T < \frac{3}{4} \times 10^{-3}$.

b)



c) The simplification in part a) yields an equivalent DT LTI system between the C/D and D/C converters. Thus the overall system will be LTI if $x_c(t)$ is appropriately bandlimited to avoid aliasing, as assumed in this part. There are no conditions on $H(e^{j\omega})$.