EECE 491: Discrete-time Signal Processing

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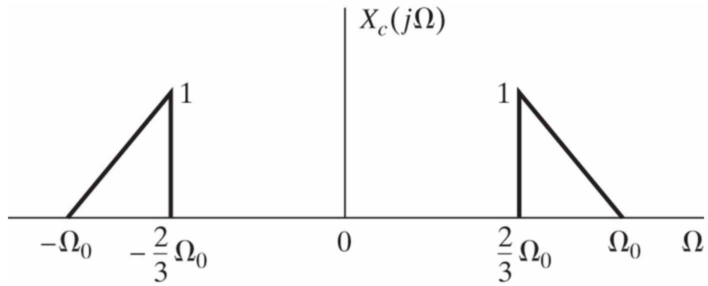
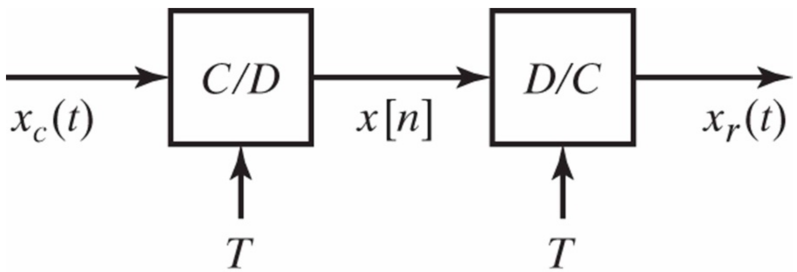
Practice Problems on
Sampling

Problem 1

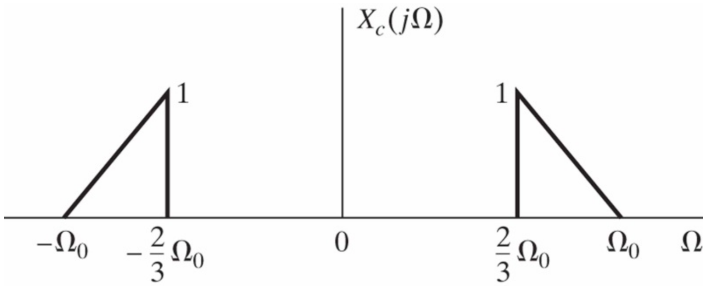
- $x_c(t)$ **below is passed through the system on the bottom. Determine the range of values of** T **for which** $x_r(t) = x_c(t)$**.**

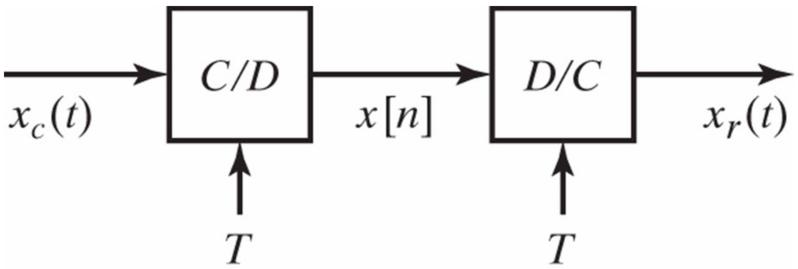




Problem 1 Solution

- $x_c(t)$ passed through the system on the right. Determine the range of values of T for which x_r(t) = $x_c(t)$.

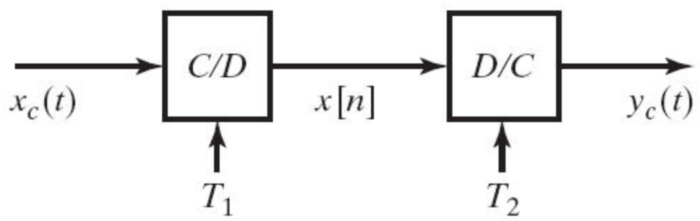




- **Solution:** $\boxed{\Omega_0 T \le \pi}$

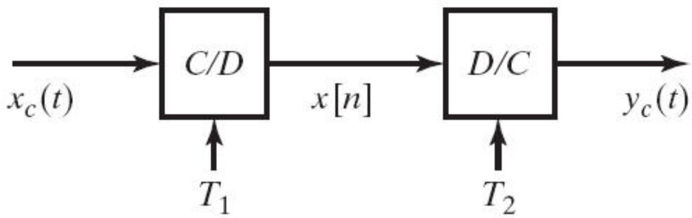
Problem 2

- **For the system below, assume** $X_c(j\Omega) = 0, \quad |\Omega| \geq \pi / T_1$ **. Express** $y_c(t)$ **in terms of** $x_c(t)$**. Is the relationship different for** $T_1 > T_2$ **and** $T_1 < T_2$**?**



Problem 2 Solution

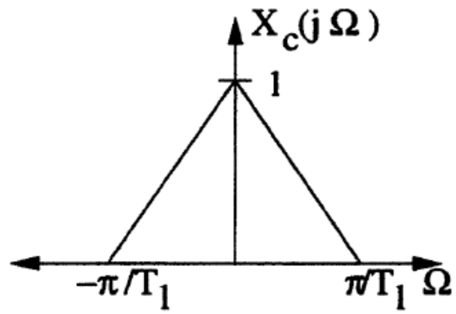
- **For the system below, assume $X_c(j\Omega)=0, \quad |\Omega| \geq \pi/T_1$. Express $y_c(t)$ in terms of $x_c(t)$. Is the relationship different for $T_1 > T_2$ and $T_1 < T_2$?**



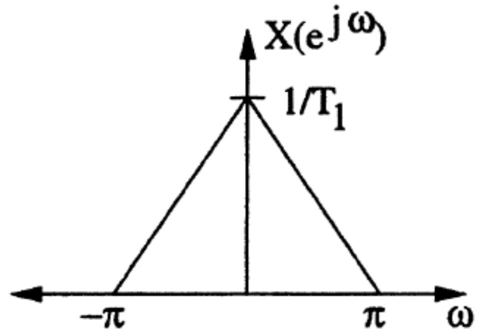
- **Solution:**

$$X_c(j\Omega)=0, \quad |\Omega| \geq \pi/T_1$$

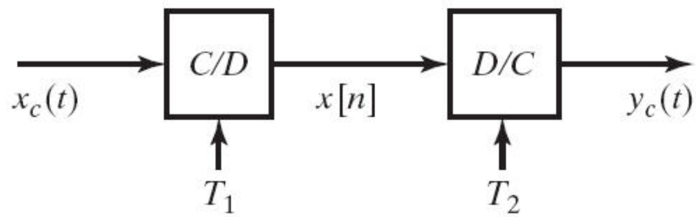
$$x[n]=x_c(nT_1)$$



$$\boxed{X\left(e^{j\omega}\right)=\frac{1}{T_1}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T_1}-\frac{2\pi k}{T_1}\right)\right)}$$



Problem 2 Solution (cont'd)



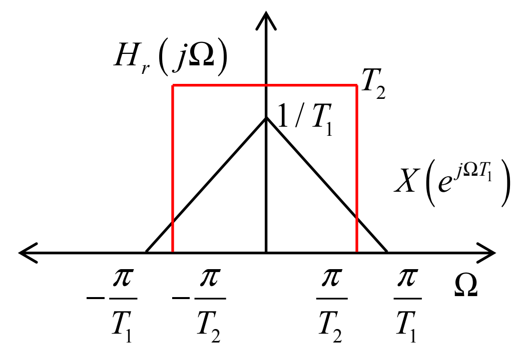
- **$y_c(t)$ is a bandlimited interpolation of $x[n]$ at a different period than T_1:**

$$Y_c(j\Omega) = H_r(j\Omega)X\left(e^{j\Omega T_2}\right)$$

$$= \begin{cases} T_2 X\left(e^{j\Omega T_2}\right) & |\Omega| < \pi / T_2 \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \frac{T_2}{T_1}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\Omega T_2}{T_1} - \frac{2\pi k}{T_1}\right)\right) & |\Omega| < \pi / T_2 \\ 0 & otherwise \end{cases}$$

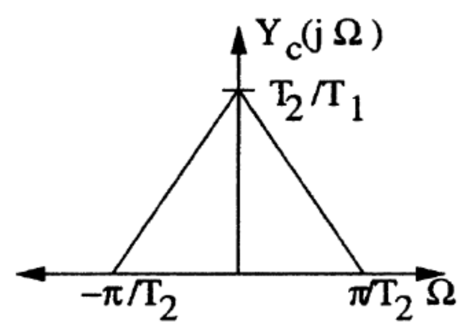
$$X\left(e^{j\omega}\right) = \frac{1}{T_1}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T_1} - \frac{2\pi k}{T_1}\right)\right)$$



$$y_c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y_c(j\Omega)e^{j\Omega t}d\Omega$$

$$= \frac{1}{2\pi}\frac{T_2}{T_1}\int_{-\pi/T_2}^{\pi/T_2}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\Omega T_2}{T_1} - \frac{2\pi k}{T_1}\right)\right)e^{j\Omega t}d\Omega$$

$$y_c(t) = \frac{T_2}{T_1}x_c\left(\frac{T_2}{T_1}t\right)$$



Problem 3

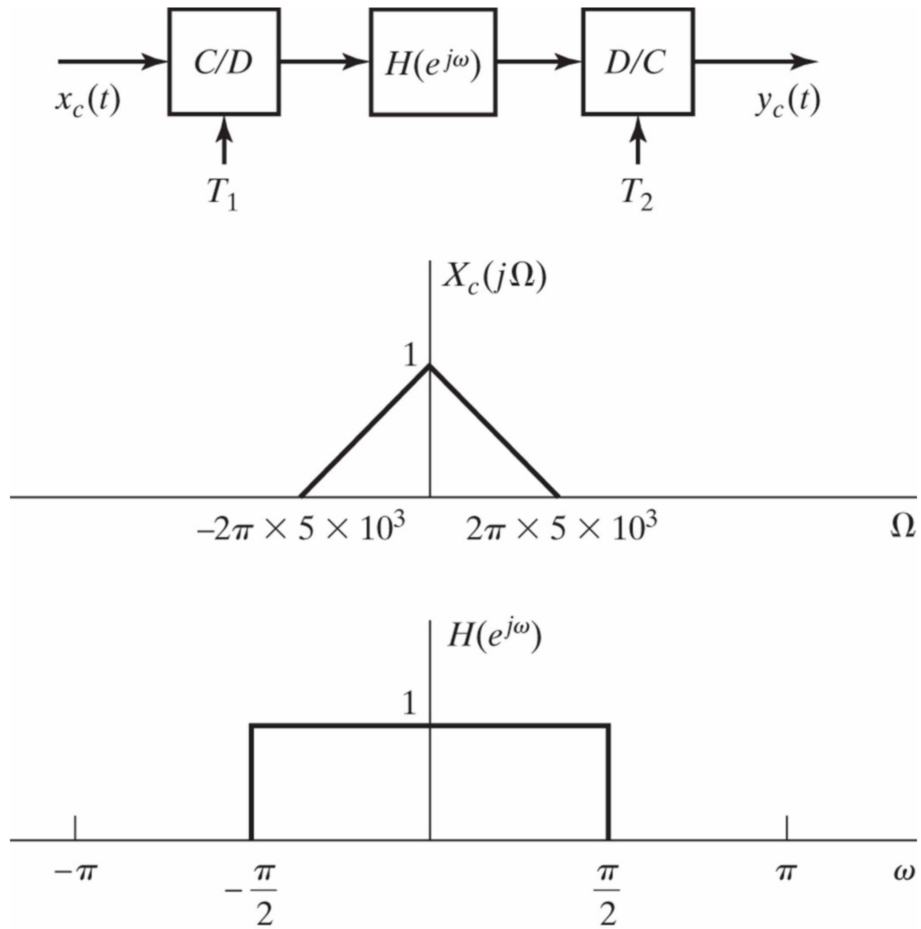
- **Consider the following system.**
- **Sketch and label** $Y_c(j\Omega)$ **when:**

a) $\dfrac{1}{T_1} = \dfrac{1}{T_2} = 10^4$

b) $\dfrac{1}{T_1} = \dfrac{1}{T_2} = 2\times 10^4$

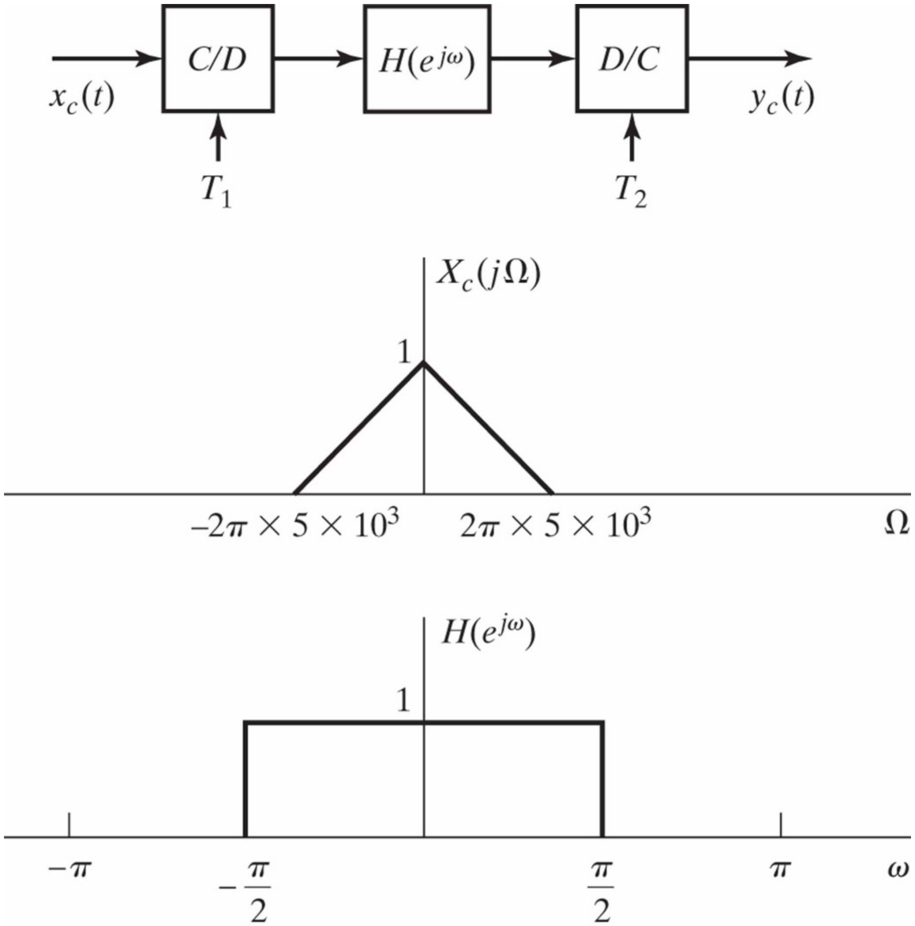
c) $\dfrac{1}{T_1} = 2\times 10^4 \qquad \dfrac{1}{T_2} = 10^4$

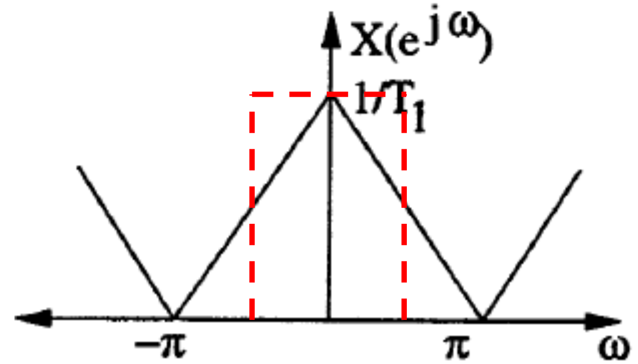
d) $\dfrac{1}{T_1} = 10^4 \qquad \dfrac{1}{T_2} = 2\times 10^4$

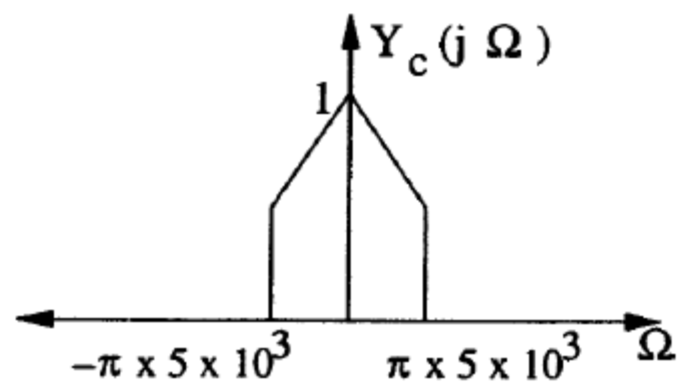


Problem 3 Solution

a) $\dfrac{1}{T_1} = \dfrac{1}{T_2} = 10^4$

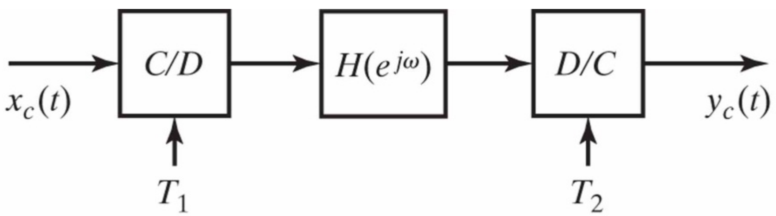


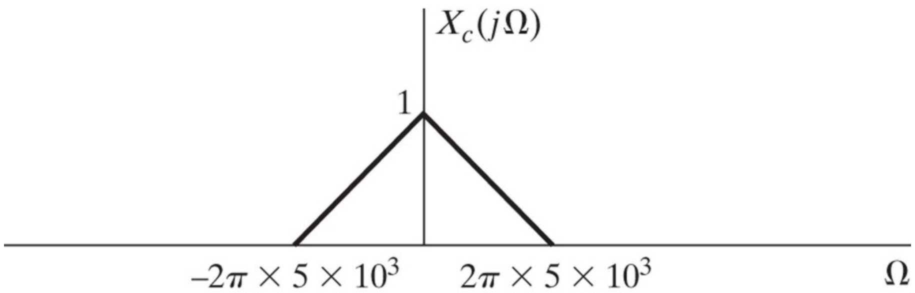


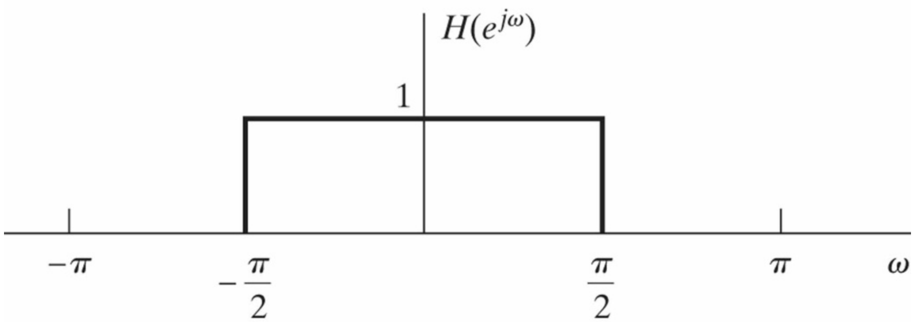


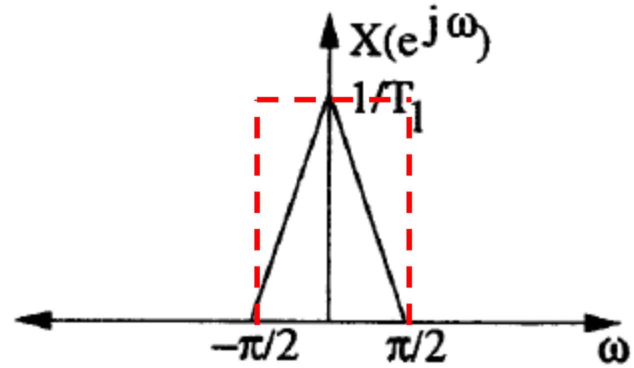
Problem 3 Solution (cont'd)

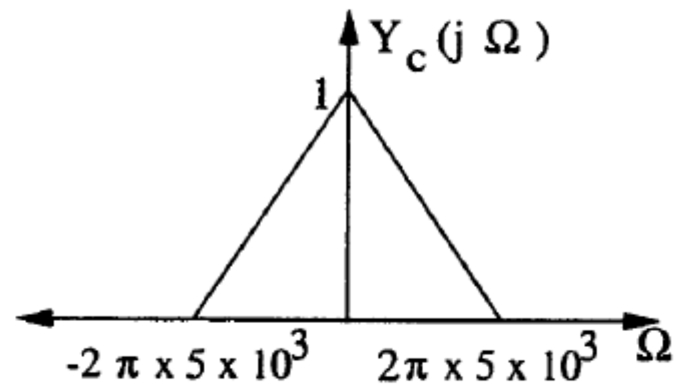
b) $\frac{1}{T_1} = \frac{1}{T_2} = 2\times10^4$





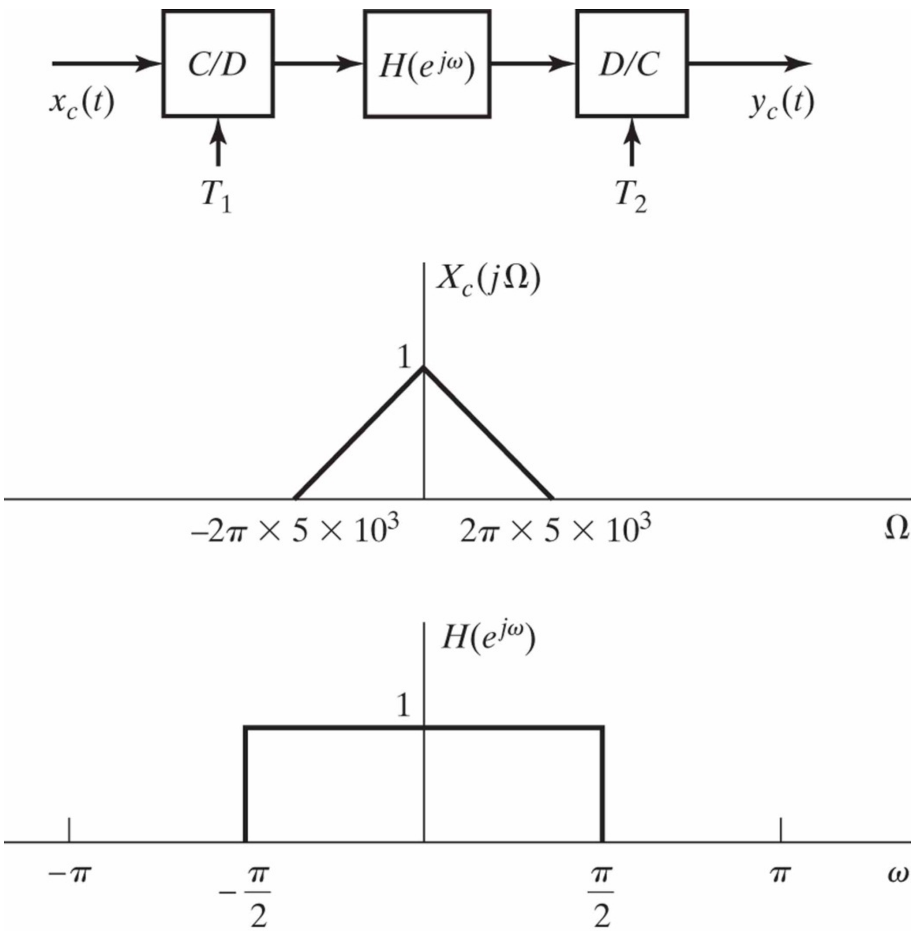


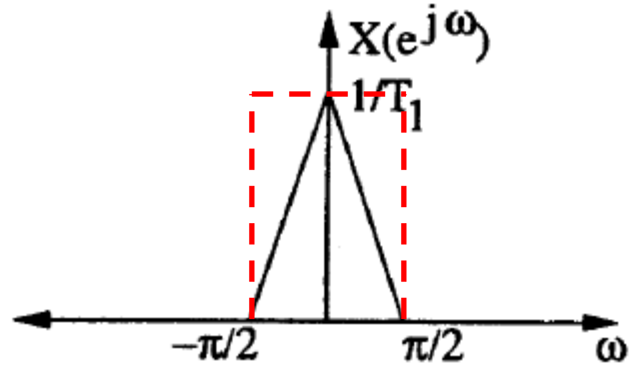


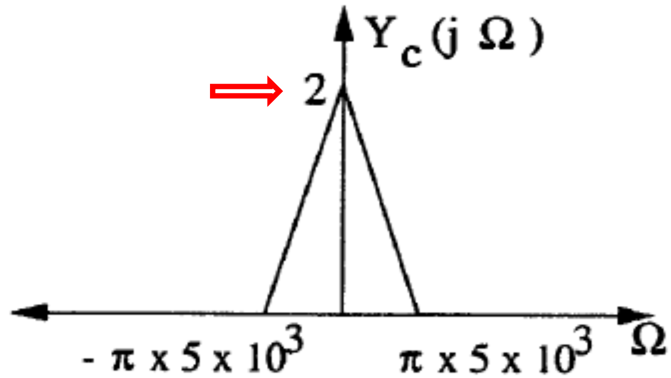


Problem 3 Solution (cont'd)

c) $\dfrac{1}{T_1} = 2\times10^4 \qquad \dfrac{1}{T_2} = 10^4$

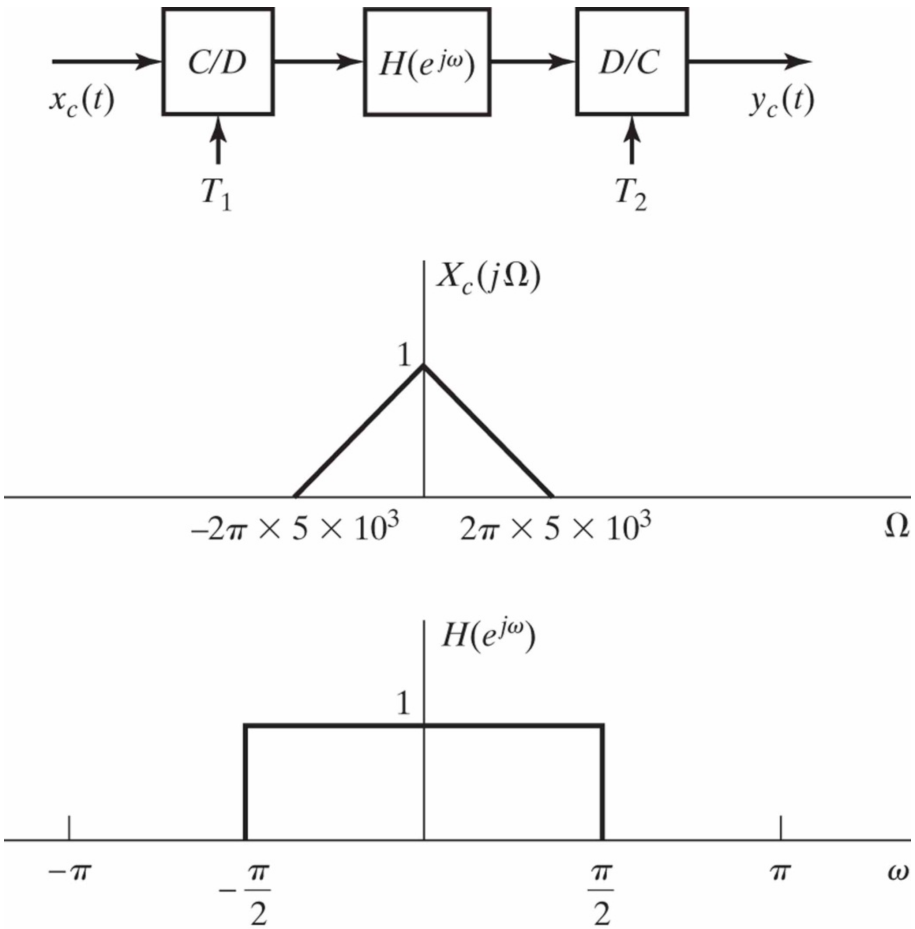


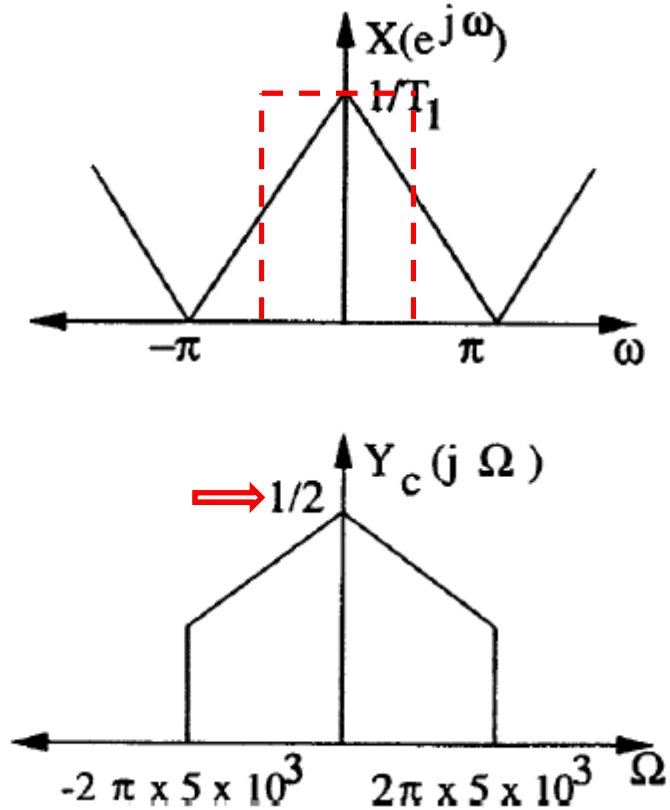




Problem 3 Solution (cont'd)

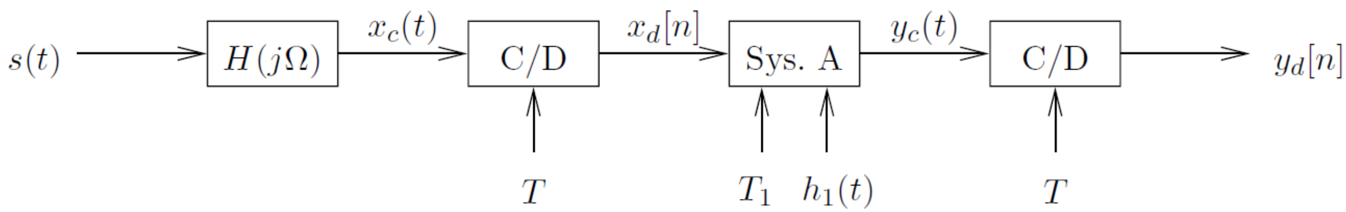
d) $\frac{1}{T_1} = 10^4 \qquad \frac{1}{T_2} = 2\times 10^4$





Problem 4

- **Consider the following system**



$H(j\Omega)$: $\quad H(j\Omega) = \begin{cases} 1, & |\Omega| < \pi \cdot 10^{-3} \text{ rad/sec} \\ 0, & |\Omega| > \pi \cdot 10^{-3} \text{ rad/sec} \end{cases}$

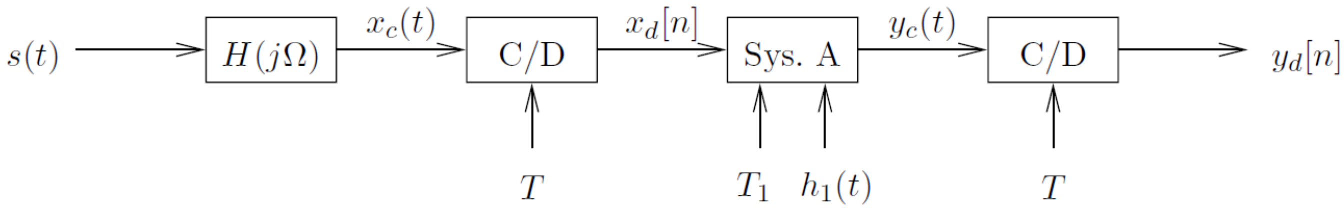
First C/D: $\quad x_d[n] = x_c(nT)$

System A: $\quad y_c(t) = \sum_{k=-\infty}^{\infty} x_d[k] h_1(t - kT_1)$

Second C/D: $\quad y_d[n] = y_c(nT)$

Problem 4 (cont'd)

- **Consider the following system**



$$H(j\Omega):\quad H(j\Omega)=\begin{cases}1, & |\Omega|<\pi\cdot 10^{-3}\ \text{rad/sec}\\ 0, & |\Omega|>\pi\cdot 10^{-3}\ \text{rad/sec}\end{cases}$$

$$\text{First C/D:}\quad x_d[n]=x_c(nT)$$

$$\text{System A:}\quad y_c(t)=\sum_{k=-\infty}^{\infty} x_d[k]h_1(t-kT_1)$$

$$\text{Second C/D:}\quad y_d[n]=y_c(nT)$$

<u>Part-1</u>: What does System A do?

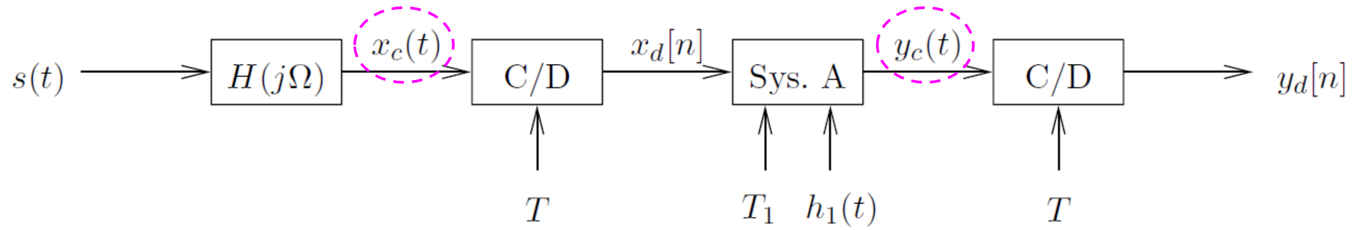
<u>Solution</u>: It converts the DT signal $x_d[n]$ into a CT signal $y_c(t)$ filtered by $h_1(t)$.

$$x_d[k] \quad\rightarrow\quad x_s(t)=\sum_{k=-\infty}^{\infty} x_d[k]\delta(t-kT_1) \quad\rightarrow\quad y_c(t)=x_s(t)*h_1(t)$$

(covert samples to impulses placed at multiples of T_1)

(convolve $x_s(t)$ with C.T. filter $h_1(t)$)

Problem 4 (cont'd)



$H(j\Omega)$: $\quad H(j\Omega) = \begin{cases} 1, & |\Omega| < \pi \cdot 10^{-3} \text{ rad/sec} \\ 0, & |\Omega| > \pi \cdot 10^{-3} \text{ rad/sec} \end{cases}$

First C/D: $\quad x_d[n] = x_c(nT)$

System A: $\quad y_c(t) = \sum_{k=-\infty}^{\infty} x_d[k] h_1(t - kT_1)$

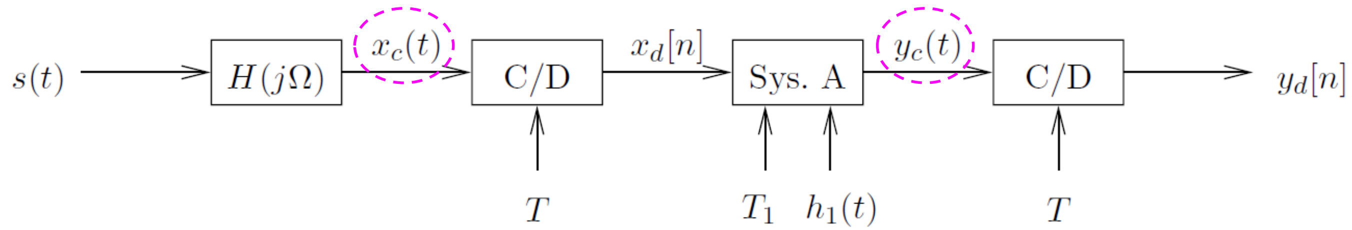
Second C/D: $\quad y_d[n] = y_c(nT)$

Part-2: Specify a choice for T, T_1 and $h_1(t)$ so that $y_c(t)$ and $x_c(t)$ are guaranteed to be equal for any choice of the input signal $s(t)$.

Solution

- $x_c(t)$ is bandlimited to $\boxed{\Omega_c = \pi \cdot 10^{-3} \text{ rad/s}}$ because $s(t)$ was filtered by $H(j\Omega)$ with that cutoff frequency
- To guarantee equality of $x_c(t)$ and $y_c(t)$, T is chosen to be sufficiently small to avoid aliasing from the first C/D converter, and System A can be an ideal D/C converter with the same sampling period.
- We have no aliasing when $\boxed{\Omega_c T < \pi \quad \Rightarrow \quad T < 1000s}$
- System A is an ideal D/C converter when $h_1(t)$ is an appropriate sinc function $\boxed{h_1(t) = \dfrac{\sin(\pi t / T)}{\pi t / T}}$
- $T_1 = T$

Problem 4 (cont'd)



$H(j\Omega)$: $\quad H(j\Omega) = \begin{cases} 1, & |\Omega| < \pi \cdot 10^{-3} \text{ rad/sec} \\ 0, & |\Omega| > \pi \cdot 10^{-3} \text{ rad/sec} \end{cases}$

System A: $\quad y_c(t) = \sum_{k=-\infty}^{\infty} x_d[k] h_1(t - kT_1)$

First C/D: $\quad x_d[n] = x_c(nT)$

Second C/D: $\quad y_d[n] = y_c(nT)$

Part-3: Is the choice in Part-2 unique for T, T_1 and $h_1(t)$ for $y_c(t)$ and $x_c(t)$ to be equal for any choice of $s(t)$, or are there other choices ?

Solution

- **Any T < 1000s works**
- **T_1 must be equal to T**
- **What about the choice of $h_1(t)$? If we choose T < 1000s, then:**

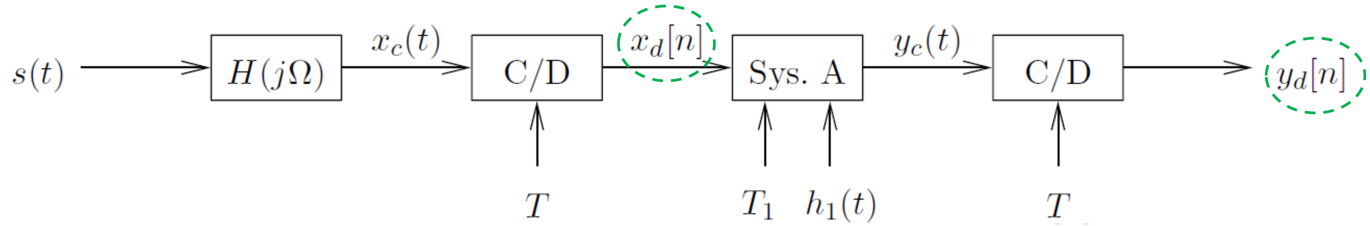
$X_d(e^{j\omega})$, DTFT of $x_d[n]$, is zero for $\frac{\pi T}{1000} < |\omega| < \pi$

$\Rightarrow X_s(j\Omega)$, CTFT of $x_s(t)$, is zero for $\frac{\pi T}{1000 T_1} = \frac{\pi}{1000} < |\Omega| < \frac{\pi}{T_1}$

$H_1(j\Omega)$, CTFT of $h_1(t)$, can be anything in that frequency range

$$H_1(j\Omega) = \begin{cases} T & |\Omega| < \pi/1000 \\ \text{anything} & \pi/1000 < |\Omega| < \pi/T \\ 0 & |\Omega| > \pi/T \end{cases}$$

Problem 4 (cont'd)



$H(j\Omega)$: $\quad H(j\Omega) = \begin{cases} 1, & |\Omega| < \pi \cdot 10^{-3} \text{ rad/sec} \\ 0, & |\Omega| > \pi \cdot 10^{-3} \text{ rad/sec} \end{cases}$

System A: $\quad y_c(t) = \sum_{k=-\infty}^{\infty} x_d[k] h_1(t - kT_1)$

First C/D: $\quad x_d[n] = x_c(nT)$

Second C/D: $\quad y_d[n] = y_c(nT)$

<u>Part-4</u>: Determine the most general conditions you can on T, T_1 and $h_1(t)$ so that y_d[n]= $x_d[n]$ (condition called consistent resampling). Consider what happens $x_d[n]$ = $[n-n_0]$ for an integer n_0.

<u>Solution</u>: System A constructs a CT signal $y_c(t)$ from $x_d[n]$, which is then resampled to obtain $y_d[n]$. The resampling is referred to as consistent if y_d[n]= $x_d[n]$.

- **System A takes each sample of $x_d[n]$ and replaces it with $h_1(t)$ delayed by nT_1 and scaled by $x_d[n]$ at that point.**
- **The C/D converter resamples the result**

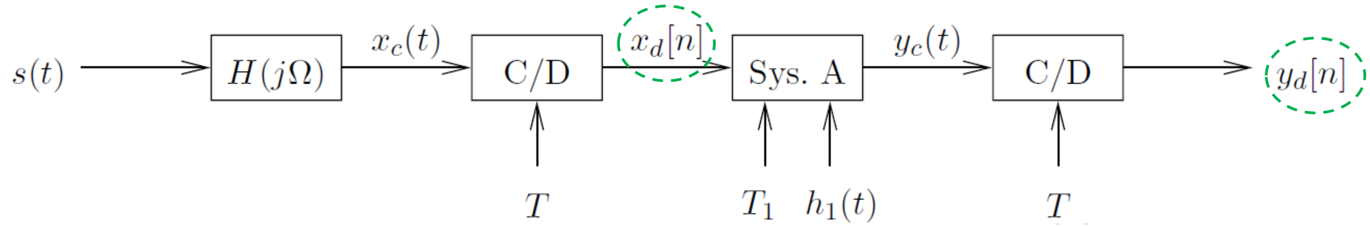
$$\boxed{x_d[n] = \delta[n-n_0] \quad \Rightarrow \quad y_c(t) = h_1(t - n_0 T_1)}$$

$$\boxed{y_d[n] = y_c(nT) = h_1(nT - n_0 T_1)}$$

$$\boxed{\text{consistent resampling } \Rightarrow h_1(nT - n_0 T_1) = \delta[n-n_0]}$$

- **Because of linearity between $x_d[n]$ and y_d[n], this is the only condition that must be checked**

Problem 4 (cont'd)



$H(j\Omega)$: $\quad H(j\Omega) = \begin{cases} 1, & |\Omega| < \pi \cdot 10^{-3} \text{ rad/sec} \\ 0, & |\Omega| > \pi \cdot 10^{-3} \text{ rad/sec} \end{cases}$

First C/D: $\quad x_d[n] = x_c(nT)$

System A: $\quad y_c(t) = \sum_{k=-\infty}^{\infty} x_d[k] h_1(t - kT_1)$

Second C/D: $\quad y_d[n] = y_c(nT)$

$$h_1(nT - n_0T_1) = \delta[n - n_0]$$

$$\Rightarrow \begin{array}{l} \text{evaluating at } n = n_0: \quad 1 = h_1(nT - nT_1) = h_1(n(T - T_1)) \\ \text{evaluating at } n \neq n_0: \quad 0 = h_1(nT - n_0T_1) \end{array}$$

The case of practical significance is when $T = T_1$. Then

$$\Rightarrow \begin{array}{l} \text{evaluating at } n = n_0: \quad 1 = h_1(0) \\ \text{evaluating at } n \neq n_0: \quad 0 = h_1(t), t = mT, m \neq 0 \end{array}$$

Therefore, $h_1(t)$ should satisfy an interpolating condition: $h_1(0) = 1$ and $h_1(t) = 0$ for all multiples of T.

It doesn't matter what $h_1(t)$ is for other values of t.