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EECE 691 – Digital Signal Processing

Problem Set 1: Basics of Signals

Problem 1: Review on Continuous-time Differential Equations

Solve the following differential equation for $y(t)$

$$y(t)+6\dot{y}(t)+5\ddot{y}(t)=1$$

for $t \geq 0$, assuming the initial conditions $y(0)=1$ and $\dot{y}(0)=1$. Express your solution in closed form. Assume the homogeneous solution has the form $Ae^{s_1 t}+Be^{s_2 t}$. Can you explain mathematically why we need to specify two initial conditions on $y(t)$?

Problem 2: Geometric Sums

a) Express the sum $\sum_{n=0}^{N}\alpha^n$ in closed form. For what values of α does the sum converge?

b) Expand $\frac{1}{1-\alpha}$ as a power series in terms of α by considering part a) and letting N go to ∞. For what values of α does your series converge?

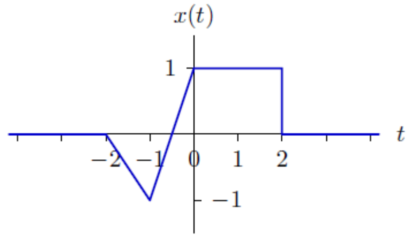
c) Can you expand $\left(\frac{1}{1-\alpha}\right)^2$ as a power series in terms of α? If so, for what values of α does your series converge? Trying using long division or Taylor series expansion.

Notes:

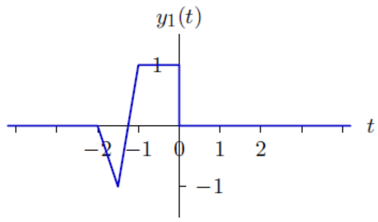
- A *sequence* is an ordered list of numbers (integers, rationals, reals, complex numbers, etc.). Sequences can be finite or infinite.
- A *series* is the **sum** of terms of a sequence. When the sequence is infinite, we talk about convergence of the corresponding series.
- A **power series** is a special form of a series that takes the form of an infinite polynomial
$$\sum_{n=0}^{\infty}a_n(x-c)^n=a_0+a_1(x-c)+\cdots+a_n(x-c)^n+\cdots$$
Here c is any real number and a series of this form is called a power series centered at c. Note that $c = 0$ is ok, and then the power series will look like $a_0+a_1x+\cdots+a_nx^n+\cdots$ Note also that a power series is just like any other series except that it depends on x.
- When a power series converges, we can define a function by this power series as $f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n$. Finding the values of x for which the power series converges determines the region of convergence of the series. The ROC can take one of three forms: 1) converges only at $x=c$, or 2) converges for all x, or 3) there is a number R called the Radius of Convergence such that the series converges for all $c-R<x<c+R$ and the series diverges outside this interval.
- If you start with a function $f(x)$ and want to find the power series representation for it, there is a nice formula for that, called the **Taylor Series**. If $f(x)$ is represented by a power series centered at c, then
$$\begin{aligned}f(x)&=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n\\&=f(c)+f'(x)\big|_{x=c}\times(x-c)+\frac{f''(x)}{2}\bigg|_{x=c}\times(x-c)^2+\cdots\end{aligned}$$
- If $c=0$, the Taylor series is called the **Maclaurin series**.

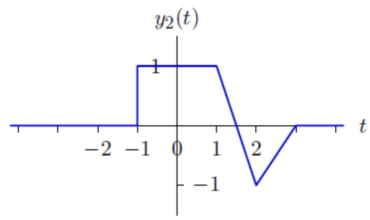
Problem 3: Continuous-time Transformations

Let $x(t)$ represent the signal shown in the following plot. The signal is zero outside the range $-2 < t < 2$.



The plots below show two functions $y_1(t)$ and $y_2(t)$, which are signals derived from $x(t)$. Determine an expression for $y_1(t)$ and $y_2(t)$ in terms of $x(t)$.





Problem 4: Convolution Integral

The convolution integral occurs frequently in physical sciences, engineering, and mathematics. The convolution integral of two functions $f_1(t)$ and $f_2(t)$ is defined as

$$f(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

The integral is performed with respect to τ and not to *t*. Evaluating the convolution first requires time-inversion of $f_2(\tau)$ to obtain $f_2(-\tau)$, then shifting by the amount *t* at which we are interested in viewing the convolution result. Notice that in this "mixing" operation, although we might be interested in determining the result at one instance in time, *t*, the values of the functions $f_1(t)$ and $f_2(t)$ at *all* values in time are required.

Can you explain why the convolution integral is defined as above and not as

$$\int_{-\infty}^{\infty} f_1(\tau) f_2(\tau - t) d\tau$$

where time inversion is done on t rather than on τ ?

Note: For a history on the Convolution Operation, read **"A History of the Convolution Operation" by A. Dominguez in *IEEE Pulse Magazine*, Jan. 2015**. URL: http://pulse.embs.org/january-2015/history-convolution-operation/.

Problem 5: An analog signal $x_a(t) = \cos(480\pi t) + 3\cos(720\pi t)$ is sampled 600 times per second to obtain the discrete-time signal $x[n]$.

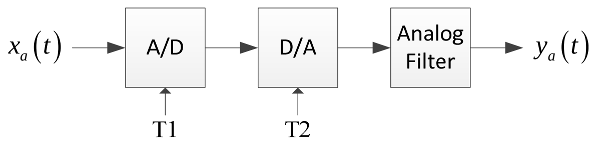
a) What is the Nyquist frequency of $x_a(t)$?

b) What frequencies appear in $x[n]$?

c) Assume now we want to transmit $x[n]$ over a digital communication link that carries binary-coded words representing the samples in $x[n]$. If each sample is quantized into 512 different voltage levels, what is the transmission bit rate (in bits/s) of the communication link?

d) If $x[n]$ is passed through an ideal D/A converter, what is the reconstructed signal $y_a(t)$?

Problem 6: Consider the signal processing system shown below. The sampling periods of the A/D and D/A converters are T1 = 5ms and T2 = 1ms, respectively. Here the analog signal is simply sampled at one rate but reconstructed at another rate. The block labeled Analog Filter just removes any frequency components that are above $F_s/2$, where $F_s = 1/T1$. Determine an expression of the output $y_a(t)$ if the input is $x_a(t) = 3\sin(100\pi t) + 2\cos(250\pi t)$.



Problem 7: Consider the continuous time signal $x_a(t) = \sin(2\pi F_0 t), -\infty < t < \infty$. Its sampled version is $x[n] = x_a(nT) = \sin 2\pi F_0 / F_s n$.

a) Using Matlab, plot $x_a(t)$ for $t \geq 0$. Then plot $x[n]$ for $0 \leq n \leq 99$ for the following values of F_0: $F_0 = 0.5, 2, 3, 4.5$ kHz, when the sampling frequency is $F_s = 5$ kHz.

b) What is the frequency of the signal $x[n]$ when $F_0 = 2$kHz and $F_s = 5$kHz ?

c) Plot the signal $y[n]$ created by taking the even-numbered samples of $x[n]$ when $F_0 = 2$kHz and $F_s = 5$kHz . Is $y[n]$ sinusoidal? If so, what is it frequency?