Math 201 — Fall 2005–06
Calculus and Analytic Geometry III, sections 5–8
Quiz 1, October 27 — Duration: 1 hour

GRADES (each problem is worth 12 points):

1	2	3	4	5	6	TOTAL/72

YOUR NAME: Corrected version (i.e. with solutions) of Professor Makdisi's Quiz I, fall 2005-06

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 5	Section 6	Section 7	Section 8
Recitation Tu 11	Recitation Tu 12:30	Recitation Tu 2	Recitation Tu 3:30
Ms. Jaber	Ms. Jaber	Ms. Jaber	Professor Makdisi

INSTRUCTIONS:

1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside the booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit. Each problem is worth 12 points.
3. You may use the back of each page for scratchwork OR for solutions. There are three extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. Open book and notes. NO CALCULATORS ALLOWED. Turn OFF and put away any cell phones.

GOOD LUCK!

An overview of the exam problems. Each problem is worth 12 points.
Take a minute to look at all the questions, THEN
solve each problem on its corresponding page INSIDE the booklet.

1. (3 pts for each part, total 12 pts) Which of the following series converge, and why?

a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{\ln n}}$ b) $\sum_{n=0}^{\infty} \frac{3^n + n^2}{n!}$ c) $\sum_{n=1}^{\infty} \left(\frac{3n + \ln n}{2n - \ln n}\right)^n$ d) $\sum_{n=1}^{\infty} \frac{(\sin n)(\ln n)}{n^2}$

2. (5 pts for part (a), 7 pts for part (b), total 12 pts)

a) Find ONLY the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n!(n+2)!}{(2n)!}x^n$. Do NOT test the endpoints.

b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n+(-1)^n}{3^n}x^n$, and DO test the endpoints. What is the interval of convergence?

3. (6 pts for each part, total 12 pts)

a) Find the second-order Taylor polynomial $P_2(x)$ of the function $f(x) = (4+x)^{3/2}$, at the center $a = 0$. (I.e., P_2 is the second-order Maclaurin polynomial.)

b) For $|x| \leq 0.1$, estimate the error $|f(x) - P_2(x)|$. Your answer should have the form $|f(x) - P_2(x)| \leq B$, where B is an explicit constant. You do not have to simplify your expression for B.

4. (6 pts for each part, total 12 pts)

a) Using power series, express the integral $L = \int_{x=0}^{0.1} e^{-x^3}\,dx$ as a certain alternating series. For full credit, the answer should be written using Σ notation. You can get nearly full credit for just writing out the first four (nonzero) terms in the series.

b) Find (with justification, of course) a specific partial sum s_n for which the error satisfies $|s_n - L| < 10^{-11}$.

Note: in parts (a) and (b), you may use without proof the fact that your series satisfies the conditions of the alternating series estimation theorem.

5. (6 pts for each part, total 12 pts)

a) Let $f(x) = \sin(2x^2)$. Use the Maclaurin series of f to find the value $f^{(10)}(0)$ of the tenth derivative of f at $x = 0$. There is no need to simplify your answer.

b) (UNRELATED) Using series, find $\lim_{n\to\infty} \cos^{-1}\left[n^2 \ln(1 + \frac{1}{n}) - n\right]$.

6. (total 12 pts) We are given a geometric series G and a telescoping series T as follows:

$$G = \sum_{n=1}^{\infty} \frac{1}{3^n}, \qquad T = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right].$$

Recall that the partial sums g_N and t_N of these two series are given by

$$g_N = \sum_{n=1}^{N} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^N}, \qquad t_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}.$$

a) (4 pts) Find simple formulas for g_N and t_N in terms of N.

b) (2 pts) Evaluate the sums of the series G and T.

c) (3 pts) How large must N be to guarantee that $|t_N - T| \leq 0.001$?

d) (3 pts) How large must N be to guarantee that $|g_N - G| \leq 0.001$? (Helpful hint: $3^4 = 81 > 80$. Think at home about why the answers to (c) and (d) are so different.)

1. (3 pts for each part, total 12 pts) Which of the following series converge, and why?

a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{\ln n}}$

this is an alternating series with $u_n = \frac{1}{\sqrt{\ln n}} > 0$,

- Here as n increases, $\ln n$ increases, so $\sqrt{\ln n}$ increases, so $\frac{1}{\sqrt{\ln n}}$ decreases.
- $\lim_{n\to\infty} \frac{1}{\sqrt{\ln n}} = \frac{1}{\infty} = 0$

$\therefore \sum_{2}^{\infty} \frac{(-1)^n}{\sqrt{\ln n}}$ Converges by the alternating series test.

b) $\sum_{n=0}^{\infty} \frac{3^n + n^2}{n!}$

Use poly growth <<< exp. growth <<< factorial growth

$n^2 <<< 3^n < 4^n <<< n!$ for large n

so $\frac{3^n+n^2}{n!} << \frac{3^n+3^n}{4^n} = 2\cdot\left(\frac{3}{4}\right)^n$,

but $\sum \left(\frac{3}{4}\right)^n$ converges (geom. series with $r = \frac{3}{4} < 1$)

$\therefore \sum_{0}^{\infty} \frac{3^n+n^2}{n!}$ converges by DCT

c) $\sum_{n=1}^{\infty} \left(\frac{3n + \ln n}{2n - \ln n}\right)^n$

Use root test: $\ell = \lim_{n\to\infty} \sqrt[n]{\left(\frac{3n+\ln n}{2n-\ln n}\right)^n} = \lim_{n\to\infty} \frac{3n+\ln n}{2n-\ln n}\cdot\frac{1/n}{1/n}$

$= \lim_{n\to\infty} \frac{3+\frac{\ln n}{n}}{2-\frac{\ln n}{n}} = \frac{3+0}{2+0} = \frac{3}{2}$, since $\ln n << n$ large n

thus $\ell > 1$, so $\sum_{1}^{\infty}\left(\frac{3n+\ln n}{2n-\ln n}\right)^n$ diverges

d) $\sum_{n=1}^{\infty} \frac{(\sin n)(\ln n)}{n^2}$

put $a_n = \frac{(\sin n)(\ln n)}{n^2}$

use $|\sin n| \le 1$

$|\ln n| << n^{0.1}$ for n large

$\Rightarrow |a_n| << \frac{n^{0.1}}{n^2} = \frac{1}{n^{1.9}}$. But $\sum_{1}^{\infty} \frac{1}{n^{1.9}}$ Converges (p-series with $p = 1.9 > 1$)

$\therefore \sum |a_n|$ converges by DCT

$\therefore \sum a_n$ Converges absolutely

2. (5 pts for part (a), 7 pts for part (b), total 12 pts)

a) Find ONLY the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n!(n+2)!}{(2n)!} x^n$. Do NOT test the endpoints.

Use the ratio test;

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\,(n+3)!}{(2n+2)!}\right| |x|^{n+1} \cdot \frac{\cancel{(2n)!}}{\cancel{n!}\,\cancel{(n+2)!}\,\cancel{|x|^n}}$$

$(n+1)\cancel{n!}$ $\quad (n+3)\cancel{(n+2)!}$ $\quad (2n)! \longrightarrow (2n+2)(2n+1)\cancel{(2n)!}$

$$= \frac{(n+1)(n+3)\,|x|}{(2n+2)(2n+1)} \cdot \frac{1/n}{1/n} = \frac{(1+\frac{1}{n})(1+\frac{3}{n})}{(2+\frac{2}{n})(2+\frac{1}{n})} \cdot |x| \longrightarrow \frac{1.1}{2.2}|x| = \frac{|x|}{4}$$

so $\rho = \frac{|x|}{4}$ and $\rho < 1 \iff |x| < 4$.

So Radius of convergence is $\boxed{4}$

b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n+(-1)^n}{3^n} x^n$, and DO test the endpoints. What is the interval of convergence?

Use root test; Set $|a_n| = \left|\frac{n+(-1)^n}{3^n}\right| |x|^n$,

$$\sqrt[n]{|a_n|} = \sqrt[n]{n+(-1)^n} \cdot \frac{|x|}{3}$$

Claim: $\sqrt[n]{n+(-1)^n} \xrightarrow[n\to\infty]{} 1$. Reason: $\frac{n}{2} \leq n-1 \leq n+(-1)^n \leq n+1 \leq 2n$

and $\sqrt[n]{\frac{n}{2}} \leq \sqrt[n]{n+(-1)^n} \leq \sqrt[n]{2n} = \sqrt[n]{2} \cdot \sqrt[n]{n}$

$\sqrt[n]{\frac{n}{2}} = \frac{\sqrt[n]{n}}{\sqrt[n]{2}} \longrightarrow 1.1 = 1$ $\qquad \sqrt[n]{2}\cdot\sqrt[n]{n} \to 1.1 = 1$ as $n \to \infty$

so we get $\sqrt[n]{n+(-1)^n} \xrightarrow[n\to\infty]{} 1$ by Sandwich theorem

so $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|x|}{3}$, want $\rho < 1 \iff |x| < 3$

so radius of convergence is $\boxed{3}$

Endpts: ① $x = +3$, get $\sum_0^\infty n+(-1)^n \cdot \frac{3^n}{3^n} = \sum_0^\infty n+(-1)^n$

and $n+(-1)^n \longrightarrow \infty \neq 0$

fails n-th term test

so Diverges.

② $x = -3$, get $\sum_0^\infty (n+(-1)^n)\cdot(-1)^n$. here $|a_n| = n+(-1)^n \longrightarrow \infty$

in particular $|a_n| \not\to 0$

so $a_n \not\to 0$

so Diverges by nth term test

$\therefore$ interval of convergence is $(-3,3)$

3. (6 pts for each part, total 12 pts)

a) Find the second-order Taylor polynomial $P_2(x)$ of the function $f(x) = (4+x)^{3/2}$, at the center $a = 0$. (I.e., P_2 is the second-order Maclaurin polynomial.)

n	$f^{(n)}(x)$	$f^{(n)}(0)/n!$
0	$(4+x)^{3/2}$	$4^{3/2}/0! = 8$
1	$\frac{3}{2}(4+x)^{1/2}$	$\frac{3}{2}\cdot 4^{1/2}/1! = 3$
2	$\frac{3}{4}(4+x)^{-1/2}$	$\frac{3}{4}\cdot 4^{-1/2}/2! = \frac{3}{16}$
3	$-\frac{3}{8}(4+x)^{-3/2}$ for later	

$$\text{so } P_2(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$$

$$= 8 + 3x + \frac{3}{16}x^2$$

b) For $|x| \le 0.1$, estimate the error $|f(x) - P_2(x)|$. Your answer should have the form $|f(x) - P_2(x)| \le B$, where B is an explicit constant. You do not have to simplify your expression for B.

$f(x) = P_2(x) + R_2(x)$, so $|f - P_2| = |R_2|$

where $R_2 = \frac{f'''(c)}{3!}x^3$

for some c between 0 and x

since $|x| \le 0.1$, we get $|c| \le 0.1$

$$\text{so } |R_2| = \left| \frac{-\frac{3}{8}(4+c)^{-3/2}}{3!} \right| \cdot |x|^3 = \frac{|x|^3}{16\,(4+c)^{3/2}}$$

now $|x| \le 0.1$, and $3.9 \le 4+c \le 4.1$, since $|c| < 0.1$

so $(3.9)^{3/2} \le (4+c)^{3/2} \le (4.1)^{3/2}$

use the smaller value since it's in the denominator

$$\text{so } |R_2| \le \frac{10^{-3}}{16\cdot(3.9)^{3/2}}$$

4. (6 pts for each part, total 12 pts)

a) Using power series, express the integral $L = \int_{x=0}^{0.1} e^{-x^3}\,dx$ as a certain alternating series. For full credit, the answer should be written using Σ notation. You can get nearly full credit for just writing out the first four (nonzero) terms in the series.

have $e^{u} = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{u^n}{n!}$

So $e^{-x^3} = 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{n!}$

$$\int_0^{0.1} e^{-x^3}dx = \int_0^{0.1}\left(1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \cdots\right)dx = \int_0^{0.1}\left(\sum_0^{\infty} (-1)^n \frac{x^{3n}}{n!}\right)dx$$

$$= \left(x - \frac{x^4}{4!} + \frac{x^3}{3.2!} - \frac{x^{10}}{10.3!} + \cdots\right)\Big|_0^{0.1} = \left[\sum_0^{\infty} \frac{(-1)^n x^{3n+1}}{(3n+1)n!}\right]_0^{0.1}$$

$$= 0.1 - \frac{(0.1)^4}{4} + \frac{(0.1)^7}{7.2!} - \frac{(0.1)^{10}}{10.3!} + \cdots - 0$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (0.1)^{3n+1}}{(3n+1)n!} - 0 = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (0.1)^{3n+1}}{(3n+1)\,n!}}$$

b) Find (with justification, of course) a specific partial sum s_n for which the error satisfies $|s_n - L| < 10^{-11}$.

Note: in parts (a) and (b), you may use without proof the fact that your series satisfies the conditions of the alternating series estimation theorem.

this is an alternating series, so $|L - S_n| \leq U_{n+1}$

next U_n used term

we want $U_{n+1} < 10^{-11}$

take $U_{n+1} = \frac{(0.1)^{10}}{10 \cdot 3!} = \frac{10^{-11}}{6} < 10^{-11}$

so we take $\boxed{S_n = 0.1 - \frac{(0.1)^4}{4!} + \frac{(0.1)^7}{7 \cdot 2!}}$

5. (6 pts for each part, total 12 pts)

a) Let $f(x) = \sin(2x^2)$. Use the Maclaurin series of f to find the value $f^{(10)}(0)$ of the tenth derivative of f at $x = 0$. There is no need to simplify your answer.

- Have $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots$

$$f(x) = \sin(2x^2) = 2x^2 - \frac{2^3 x^6}{3!} + \frac{2^5 x^{10}}{5!} - \frac{2^7 x^{14}}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} c_n x^n \quad \text{with } c_n = \frac{f^{(n)}(0)}{n!}$$

So $\frac{f^{(10)}(0)}{10!} = \text{coeff of } x^{10} = \frac{2^5}{5!}$

So $\boxed{f^{(10)}(0) = \frac{10! \cdot 2^5}{5!}}$

b) (UNRELATED) Using series, find $\lim_{n\to\infty} \cos^{-1}\left[n^2 \ln(1 + \frac{1}{n}) - n\right]$.

Here $\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots = u - \frac{u^2}{2} + O(u^3)$

$$\Rightarrow \ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)$$

$$\Rightarrow n^2 \ln\left(1+\frac{1}{n}\right) - n = n - \frac{1}{2} + O\left(\frac{1}{n}\right) - n$$

$$= -\frac{1}{2} + O\left(\frac{1}{n}\right)$$

$\therefore$ as $n \to \infty$, $n^2 \ln(1+\frac{1}{n}) - n \longrightarrow -\frac{1}{2}$ since $O(\frac{1}{n}) \to 0$

Now use the fact that $\cos^{-1}x$ is continuous at $x = -\frac{1}{2}$, to say $\cos^{-1}(n^2 \ln(1+\frac{1}{n}) - n) \longrightarrow \cos^{-1}(-\frac{1}{2}) = \pi - \cos^{-1}(\frac{1}{2})$

$$= \pi - \frac{\pi}{3}$$

$$= \boxed{\frac{2\pi}{3}}$$

6. (total 12 pts) We are given a geometric series G and a telescoping series T as follows:

$$G = \sum_{n=1}^{\infty} \frac{1}{3^n}, \qquad T = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right].$$

Recall that the partial sums g_N and t_N of these two series are given by

$$g_N = \sum_{n=1}^{N} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^N}, \qquad t_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}.$$

a) (4 pts) Find simple formulas for g_N and t_N in terms of N.

$$g_N = \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^N}$$

$$\frac{1}{3} g_N = \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{N+1}}$$

$$\frac{2}{3} g_N = g_N - \frac{1}{3} g_N = \frac{1}{3} - \frac{1}{3^{N+1}}$$

so $g_N = \frac{3}{2}\left(\frac{1}{3} - \frac{1}{3^{N+1}}\right)$

$$t_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right)$$

telescopes

$$= \frac{1}{1} - \frac{1}{N+1}$$

so $t_N = 1 - \frac{1}{N+1}$

b) (2 pts) Evaluate the sums of the series G and T.

- $G = \lim_{N \to \infty} g_N = \frac{3}{2}\left(\frac{1}{3} - 0\right)$, since $\frac{1}{3^{N+1}} \to 0$

so $G = \boxed{\frac{1}{2}}$

- $T = \lim_{N \to \infty} t_N = \lim_{N \to \infty} 1 - \frac{1}{N+1} = \boxed{1}$

c) (3 pts) How large must N be to guarantee that $|t_N - T| \leq 0.001$?

$$|T - t_N| = \left|1 - \frac{1}{N+1} - 1\right| = \left|-\frac{1}{N+1}\right| = \frac{1}{N+1} \overset{\text{want}}{<} \frac{1}{1000}$$

so we can take $N+1 > 1000$

so $\boxed{N \geq 1000}$ does the trick

d) (3 pts) How large must N be to guarantee that $|g_N - G| \leq 0.001$? (Helpful hint: $3^4 = 81 > 80$. Think at home about why the answers to (c) and (d) are so different.)

$$|g_N - G| = \left|\frac{3}{2}\left(\frac{1}{3} - \frac{1}{3^{N+1}}\right) - \frac{1}{2}\right|$$

$$= \left|\frac{1}{2} - \frac{1}{2 \cdot 3^N} - \frac{1}{2}\right| = \frac{1}{2 \cdot 3^N} \overset{\text{want}}{<} \frac{1}{1000}$$

so we want $500 = \frac{1000}{2} < 3^N$

Now $3^4 > 80 \Rightarrow 3^5 > 240 > 200 \Rightarrow 3^6 > 600 > 500$

$N \geq 6 \Rightarrow 3^6 \geq 500 \Rightarrow |g_N - G| < 0.001$. So $\boxed{N \geq 6}$ does the trick