EECE 491: Discrete-time Signal Processing

Mohammad M. Mansour
Dept. of Electrical and Compute Engineering
American University of Beirut

Lecture 9: Sampling of Continuous-Time Signals

Announcements

- **Reading**
  - O&S
    - Chapter 4

Outline

1. Periodic Sampling
2. Frequency Domain Representation of Sampling
3. Reconstruction of a Band-limited Signal from its Samples
4. D.T. Processing of C.T. Signals
5. C.T. Processing of D.T. Signals

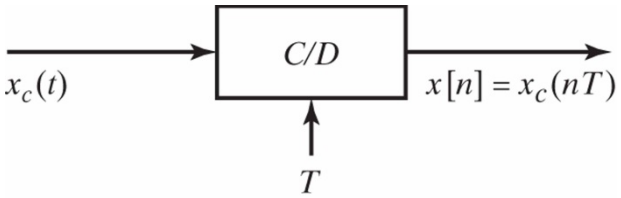
Notation

- **Continuous-time signal:** $x_c(t)$
- **Discrete-time signal:** $x[n]$
- **Continuous-time Fourier transform:** $X_c(j\Omega)$
- **Discrete-time Fourier transform:** $X\left(e^{j\omega}\right)$
- **Continuous-time frequency:** Ω
- **Discrete-time frequency:** ω
- **D.T. : Discrete-Time**
- **C.T. : Continuous-Time**
- **F.T. : Fourier Transform**

Periodic Sampling

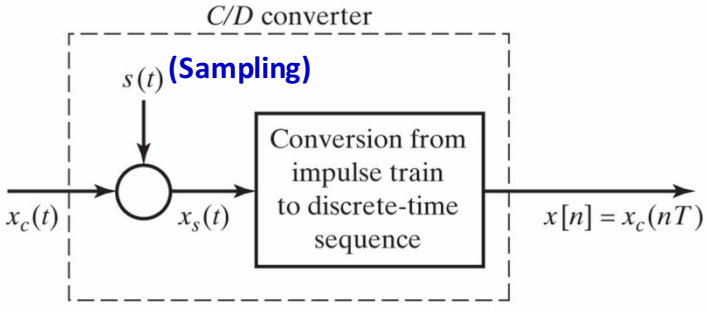
Periodic (Uniform) Sampling

- **Ideal continuous-to-discrete-time (C/D) converter**



T: Sampling period
F_s = 1/T: Sampling frequency

- **Implemented by an analog-to-digital converter: Two steps**
  - Sampling
  - Conversion of "impulse train" into a "sequence"



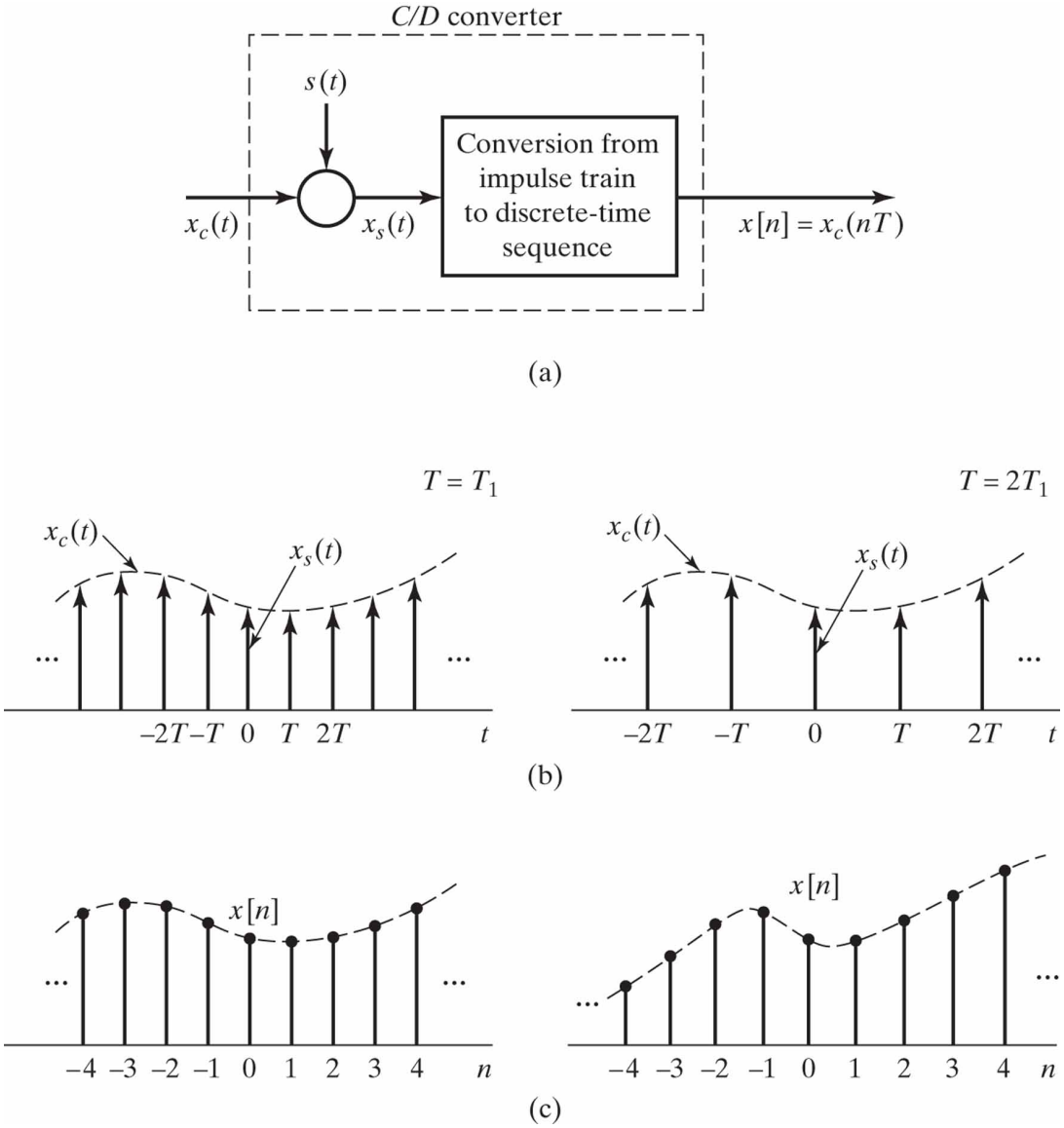
$\delta(t)$: Dirac-delta function

Impulse train: $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$

$$x_s(t) = x_c(t)s(t)$$
$$= x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$
$$= \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT)$$
$$= \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

(Area of impulse at *nT* equals value of CT signal at *nT*)

Periodic Sampling



Frequency Domain Representation of Sampling

- **F.T. $S(j\Omega)$ of periodic impulse train $s(t)$:**

$$s(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT) \xrightarrow[\Omega_S=2\pi/T]{F.T.} S(j\Omega)=\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\Omega-k\Omega_s)$$

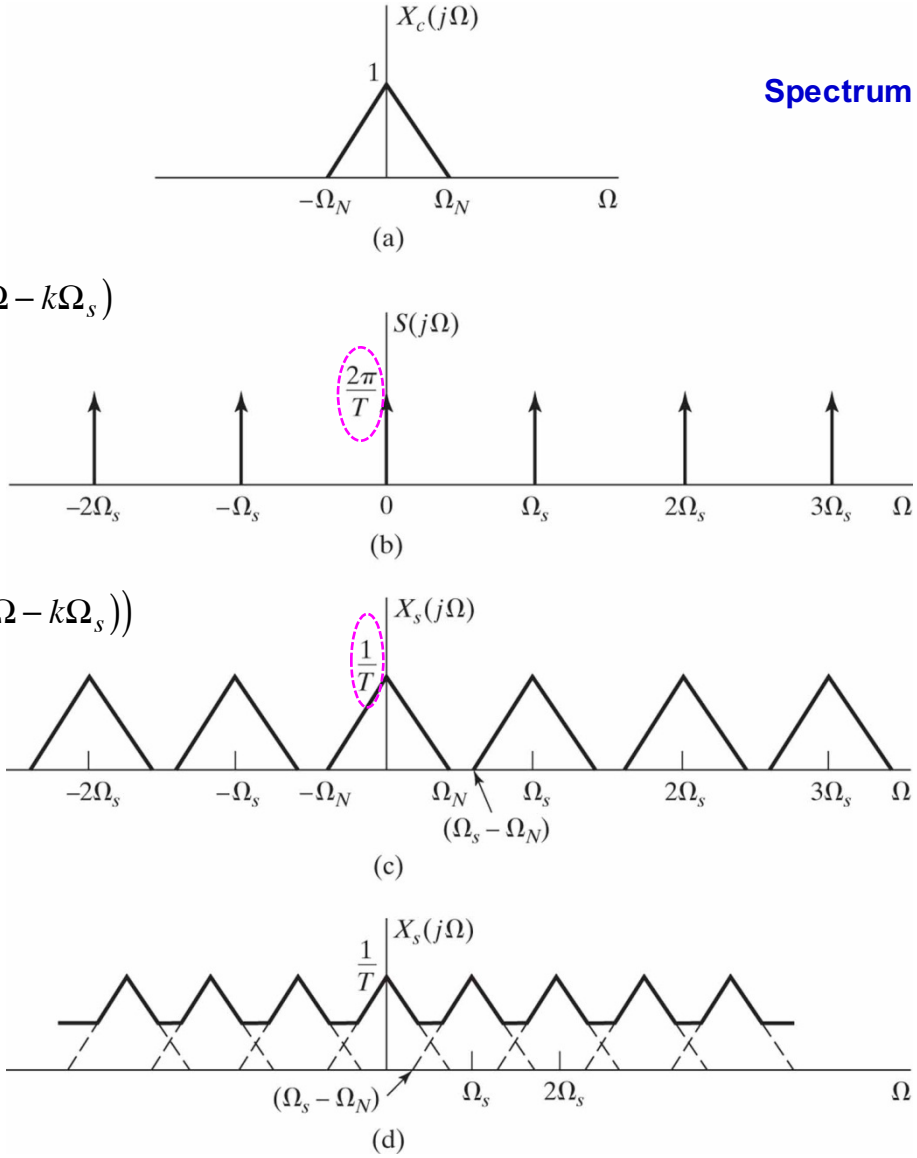
- **Sampled signal:**

$$x_s(t)=x_c(t)s(t) \xrightarrow{F.T.} X_s(j\Omega)=\frac{1}{2\pi}X_c(j\Omega)*S(j\Omega)$$

$$=\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c\left(j(\Omega-k\Omega_s)\right)$$

  - F.T. $X_s(j\Omega)$ of sampled signal consists of periodically repeated copies of $X_c(j\Omega)$
    - Period of replication is Ω_s.

Frequency Domain Representation of Sampling



Spectrum of the original signal

$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

Fourier transform of the sampling function

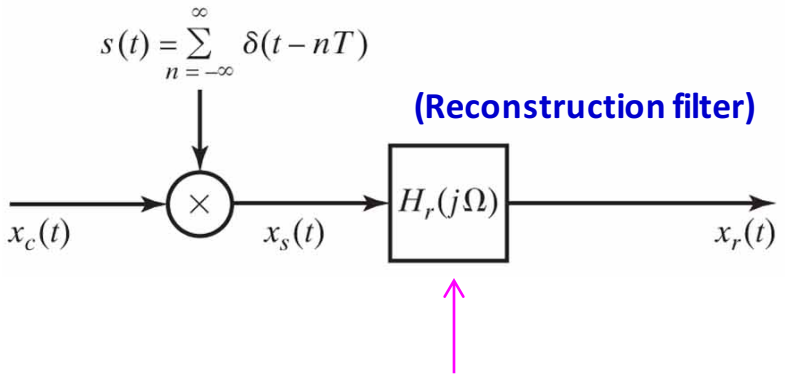
$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_s\left(j(\Omega - k\Omega_s)\right)$$

Fourier transform of the sampled signal with $\Omega_s > 2\Omega_N$.

Fourier transform of the sampled signal with $\Omega_s < 2\Omega_N$.

Frequency Domain Representation of Sampling

- When $\Omega_s > 2\Omega_N$, replicas of $X_c(j\Omega)$ do not overlap. Else, aliasing occurs.
- Hence $x_c(t)$ can be recovered from $x_s(t)$ with an idea low-pass filter
  - Also called "reconstruction filter"



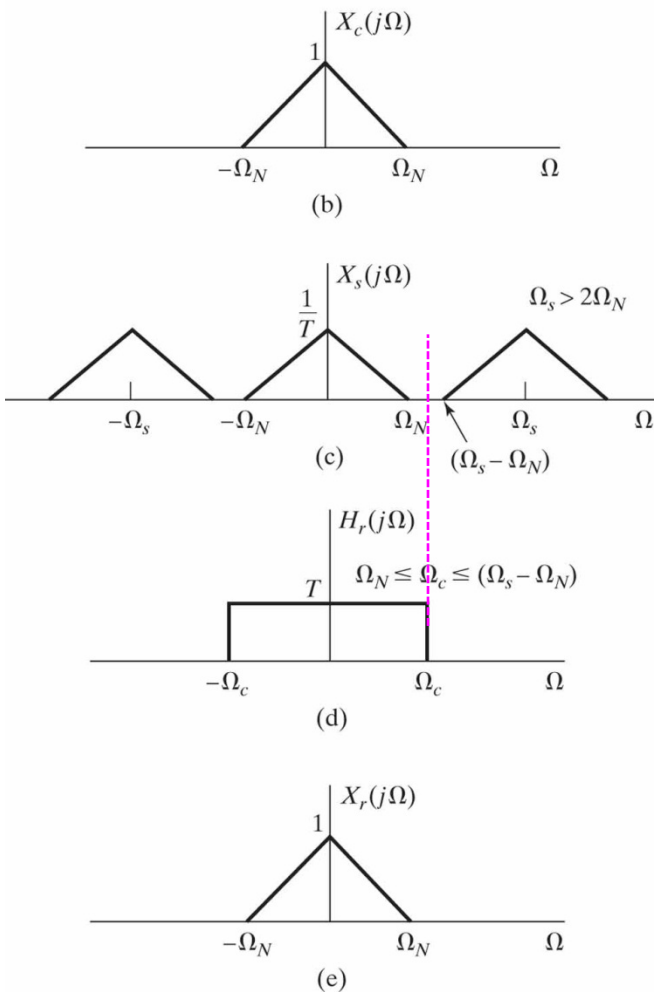
- Has cutoff frequency Ω_c satisfying

$$\Omega_N \le \Omega_c \le (\Omega_s - \Omega_N)$$

- Has gain T

$$X_r(j\Omega) = H_r(j\Omega) X_s(j\Omega)$$

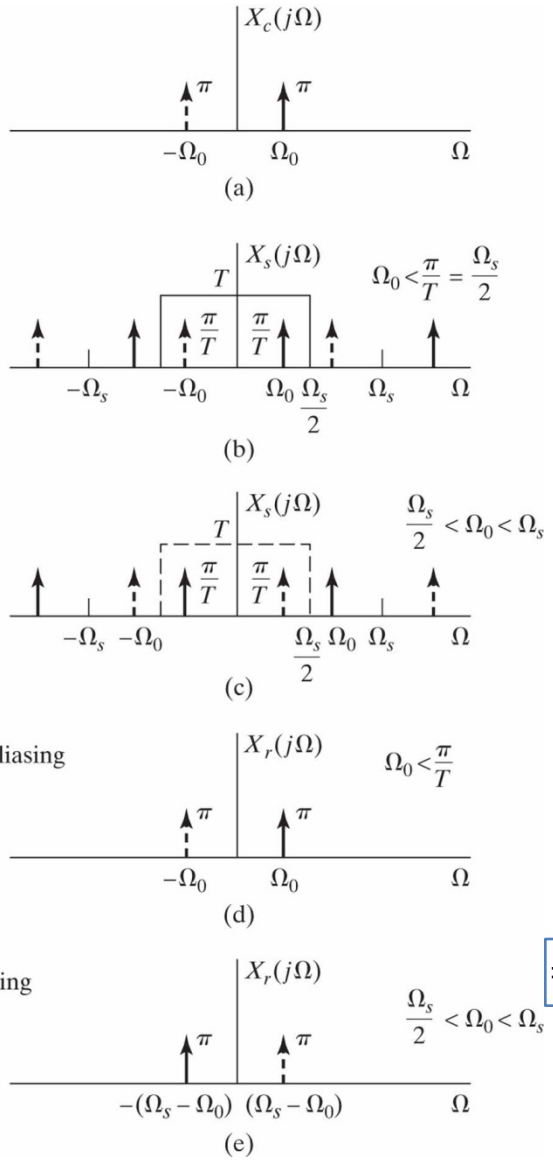
$$\Rightarrow X_r(j\Omega) = X_c(j\Omega)$$



Example 1: Impact of Aliasing

$$x_c(t) = \cos\Omega_0 t$$

$$X_c(j\Omega) = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$



Two cases:

$\Omega_s > 2\Omega_0$

$\Omega_s < 2\Omega_0$

$$\Rightarrow x_r(t) = x_c(t) = \cos\Omega_0 t$$

$$\Rightarrow x_r(t) = x_c(t) = \cos(\Omega_s - \Omega_0)t$$

Frequency Domain Representation of Sampling

Frequency Domain Representation of Sampling

- **Relationship between**

$$X_c(j\Omega) = \text{C.T.F.T}(x_c(t)) \quad ? \quad X(e^{j\omega}) = \text{D.T.F.T}(x[n])$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$$

C.T.F.T. ↓

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn}$$

But: $x[n] = x_c(nT)$ $\qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Then $X_s(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T} = X(e^{j\Omega T})$

We also proved $X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$

$$\Rightarrow X(e^{j\Omega T}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad \text{OR} \quad \Rightarrow X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

Frequency Domain Representation of Sampling

- **$X(e^{j\omega})$ is a frequency-scaled version of $X_s(j\Omega)$**
- **Frequency scaling specified by $\omega = \Omega T$**
- **Equivalent to frequency normalization:**
  - Frequency Ω_s in $X_s(j\Omega)$ normalized to $\omega = 2\pi$ in $X(e^{j\omega})$

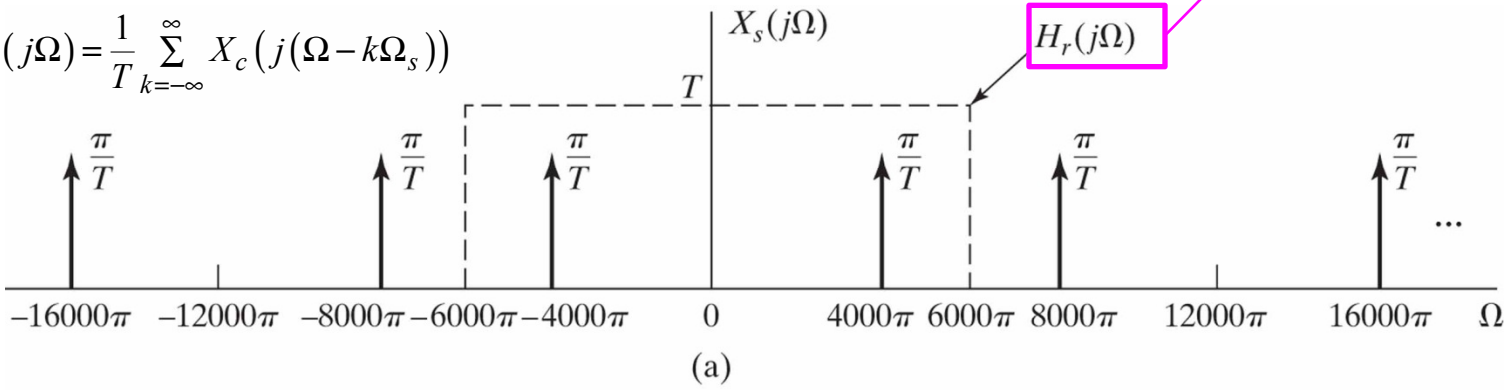
Example 2

- **Sample the CT signal $x_c(t) = \cos 4000\pi t$ at T = 1/6000 and show F.T. of CT & DT signals**
- **Solution:**
  - Highest frequency in C.T. signal Ω_N = 4000
  - Nyquist rate Ω_S = 12000π > 2Ω_N = 8000
  - No aliasing
  - F.T. $X_c(j\Omega) = \pi\delta(\Omega - 4000\pi) + \pi\delta(\Omega + 4000\pi)$

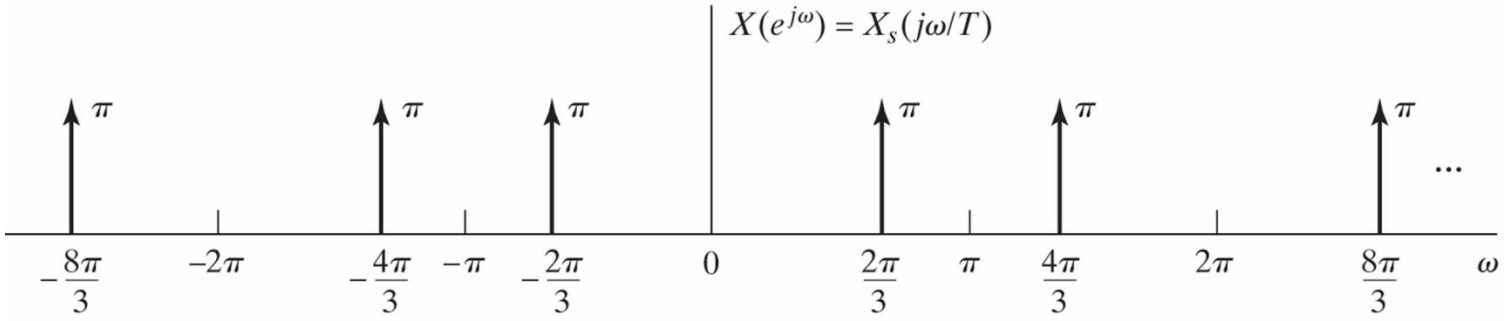
Has gain T
Has cutoff frequency Ω_c satisfying

$$\Omega_c = \Omega_s / 2$$

$$\Rightarrow X_s(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$



(a)



Example 2 (cont'd)

- **Repeat for** $x_c(t) = \cos 16000\pi t$
  - Nyquist rate not satisfied since $\Omega_S = 12000\pi < 2\,\Omega_N = 32000\pi$
  - Aliasing occurs

Reconstruction of a Band-limited Signal from its Samples

Reconstruction of a Band-limited Signal from its Samples

- Given a sequence of samples $x[n]$, can we recover the original signal $x_c(t)$?
- First form an impulse train $x_s(t)$ in which successive samples are assigned an area equal to $x[n]$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)$$

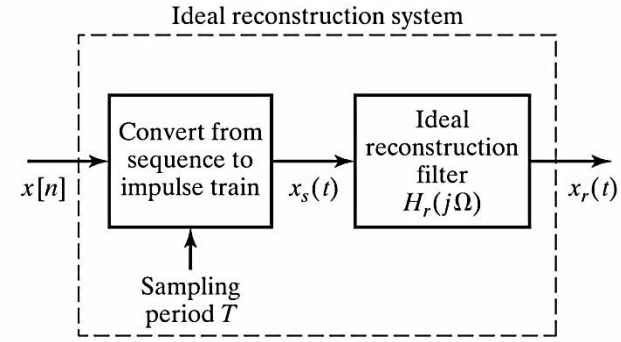
- Pass this signal thru an ideal low-pass C.T. filter with frequency response $H_r(j\Omega)$ and impulse response $h_r(t)$

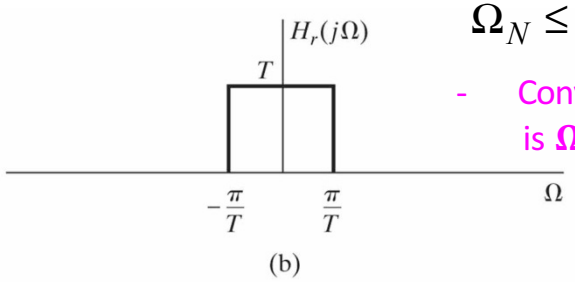
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT)$$

- $H_r(j\Omega)$ has gain T
- $H_r(j\Omega)$ has cutoff frequency Ω_c satisfying

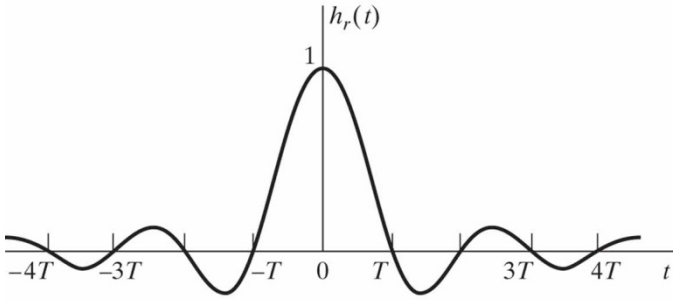
$$\Omega_N \le \Omega_c \le (\Omega_s - \Omega_N)$$

- Convenient choice of Ω_c is $\Omega_s/2$





(b)



(c)

Reconstruction of a Band-limited Signal from its Samples

- **The impulse response of the ideal LPF with cutoff frequency $\Omega_c = \Omega_s/2 = \pi/T$ is:**

$$h_r(t) = \frac{\sin(\pi t / T)}{\pi t / T}$$

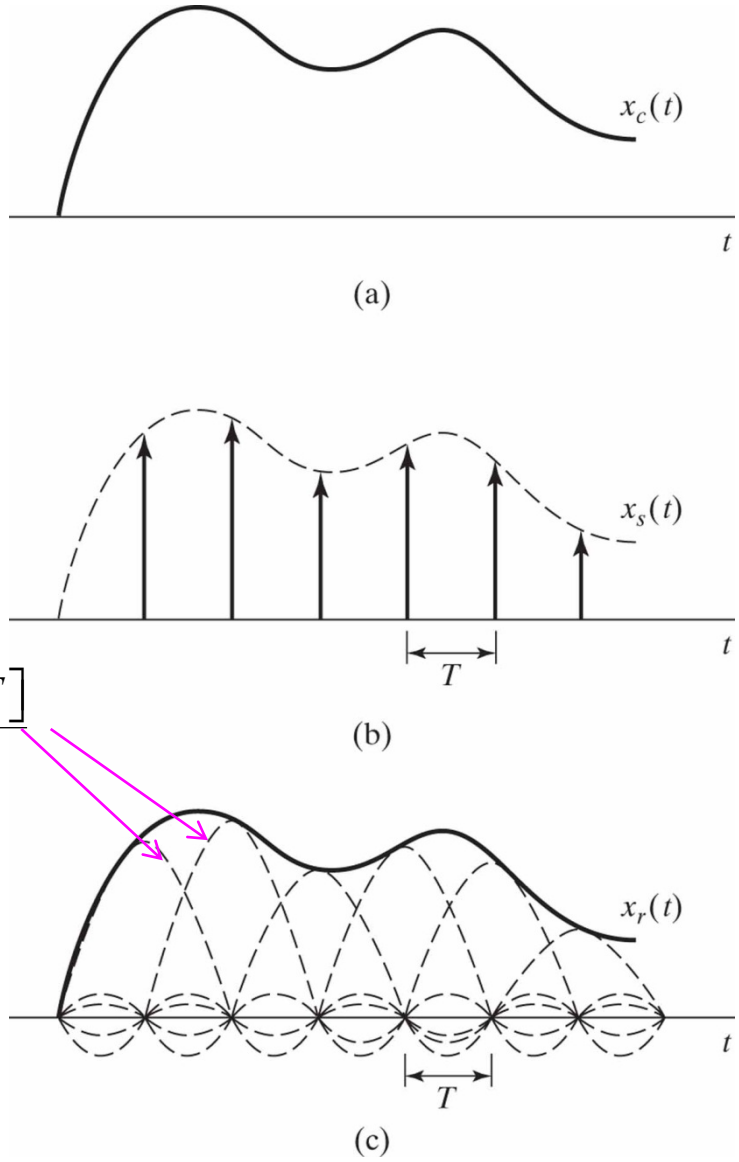
- **Then:**

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

- **Note that:**
  - $h_r(0) = 1$
  - $h_r(nT) = 0$, for n = 1, 2, …
  - Then if $x[n] = x_r(nT)$ ➔ $x_r(mT) = x_c(mT)$ for all integer values of m.
  - Therefore, reconstructed signal $x_r(t)$ has same values as the original signal $x_c(t)$ at the sampling times, independently of T.

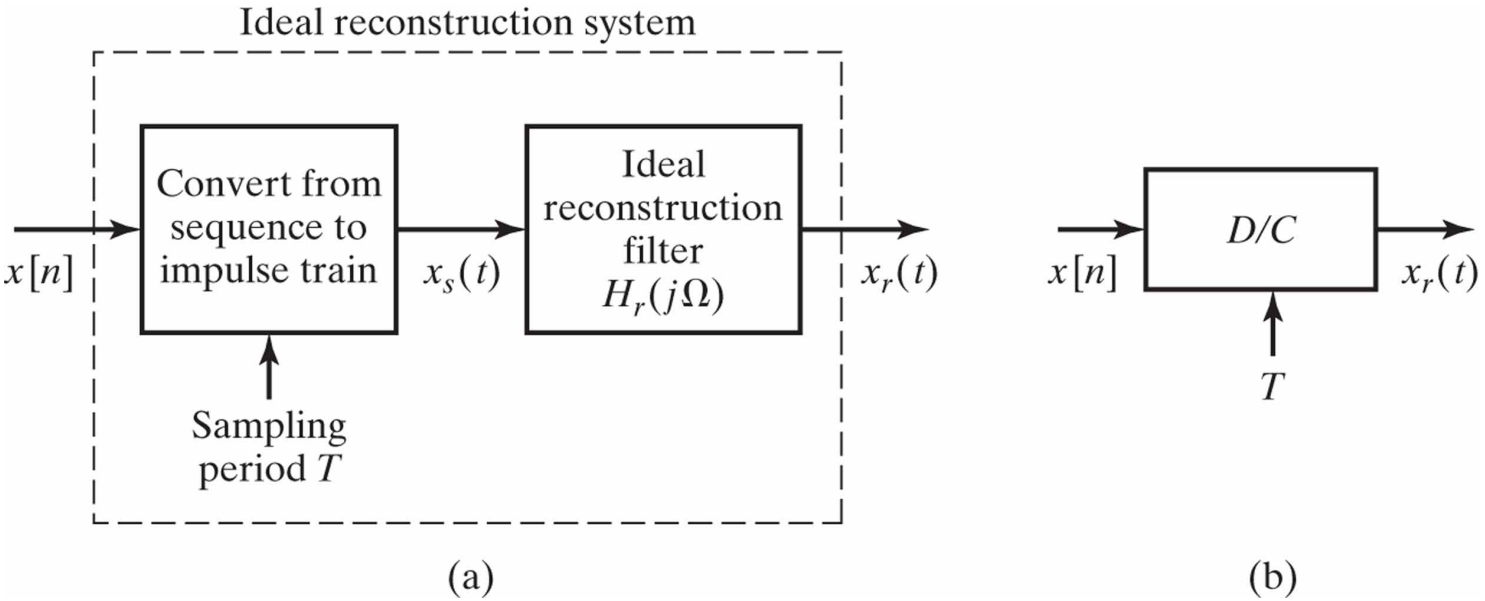
- **The ideal LPF "interpolates" between the impulses $x_s(t)$ to construct $x_c(t)$**

Reconstruction of a Band-limited Signal from its Samples



$$x[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

Ideal Discrete-to-Continuous-Time (D/C) Converter



In frequency domain:

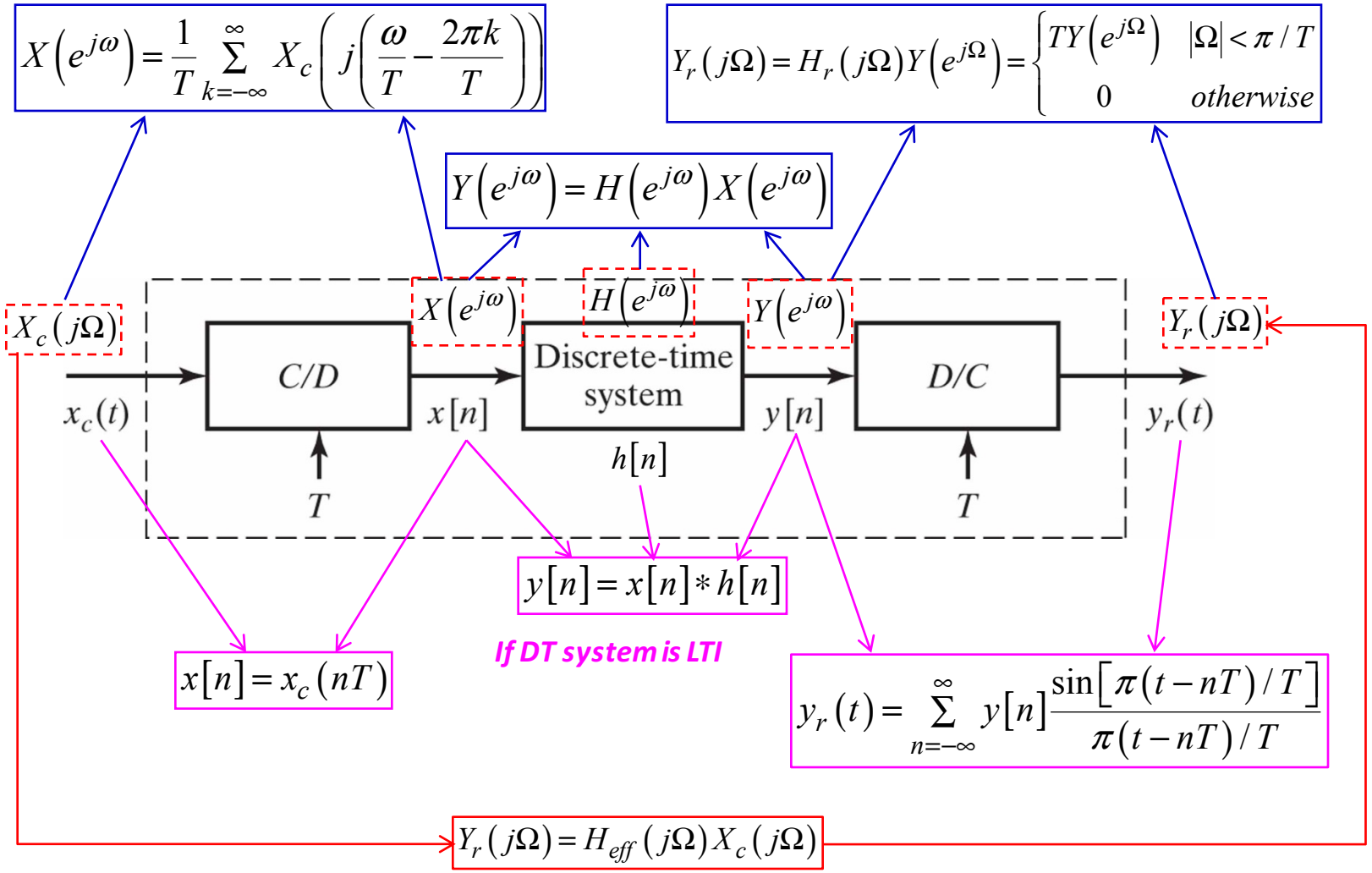
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT) \quad \Rightarrow \quad X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] H_r(j\Omega) e^{-j\Omega Tn}$$

$$= H_r(j\Omega) \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega Tn}$$

$$= H_r(j\Omega) X\left(e^{j\Omega T}\right)$$

D.T. Processing of C.T. Signals

D.T. Processing of C.T. Signals



D.T. Processing of C.T. Signals: D.T. System is LTI

- **Overall C.T. system is equivalent to an LTI system whose effective frequency response is**

$$Y_r(j\Omega) = H_{eff}(j\Omega) X_c(j\Omega)$$

  - Conditions:
    - Input signal is band-limited + sampling rate > Nyquist rate (to avoid aliasing)
    - C/D is ideal
    - D.T. system is LTI
    - D/C is ideal

- **Proof:**

$$\text{D.T. system is LTI} \Rightarrow Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega})$$

$$\text{D/C is ideal} \Rightarrow Y_r(j\Omega) = H_r(j\Omega) Y(e^{j\Omega T}) = H_r(j\Omega) H(e^{j\Omega T}) X(e^{j\Omega T})$$

Relating $X(e^{j\Omega T})$ to $X_c(j\Omega)$ thru sampling

$$Y_r(j\Omega) = H_r(j\Omega) H(e^{j\Omega T}) \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\Omega - \frac{2\pi k}{T}\right)\right)$$

$$\text{input bandlimited} \Rightarrow X_c(j\Omega) = 0 \text{ for } |\Omega| > \pi/T \quad + \quad \text{ideal } H_r(j\Omega) \text{ with cutoff } \Omega_c = \pi/T, \text{ gain } 1/T$$

$$Y_r(j\Omega) = \begin{cases} H(e^{j\Omega}) X_c(e^{j\Omega}) & |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases}$$

$$Y_r(j\Omega) = H_{eff}(j\Omega) X_c(j\Omega)$$

$$H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega}) & |\Omega| < \pi/T \\ 0 & |\Omega| \geq \pi/T \end{cases}$$

Example 3

- **Consider a discrete-time LTI system with**

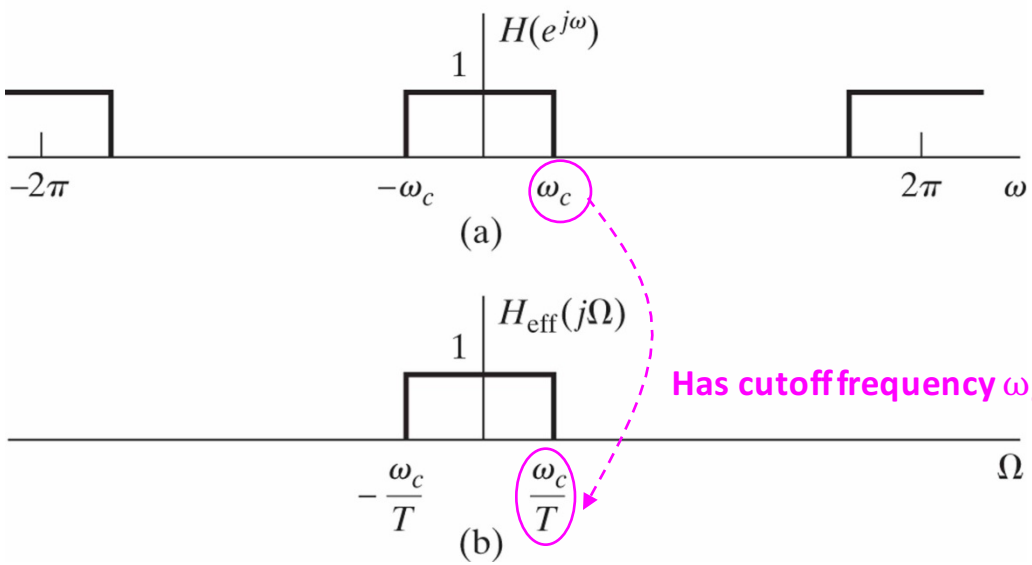
$$H\left(e^{j\omega}\right)=\begin{cases}1 & |\omega|<\omega_c\\ 0 & |\omega|\geq\omega_c\end{cases}$$

- **For band-limited inputs sampled above the Nyquist rate, the overall C.T. system will behave as an LTI C.T. system with frequency response**

$$H_{eff}\left(j\Omega\right)=\begin{cases}1 & |\Omega T|<\omega_c \text{ or } |\Omega|<\omega_c/\mathrm{T}\\ 0 & |\Omega T|\geq\omega_c \text{ or } |\Omega|\geq\omega_c/\mathrm{T}\end{cases}$$

D.T. Filter

C.T. Filter



periodic

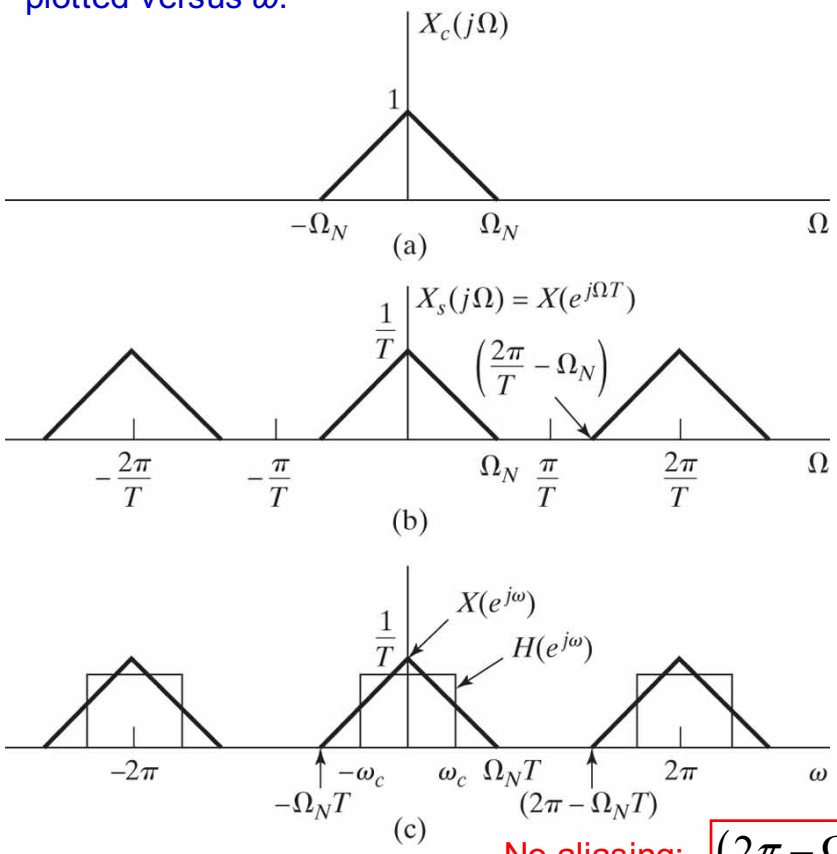
Has cutoff frequency ω_c/T

Example 3 (cont'd)

(a) F.T. of a band-limited input signal.

(b) F.T. of sampled input plotted as a function of C.T. frequency Ω.

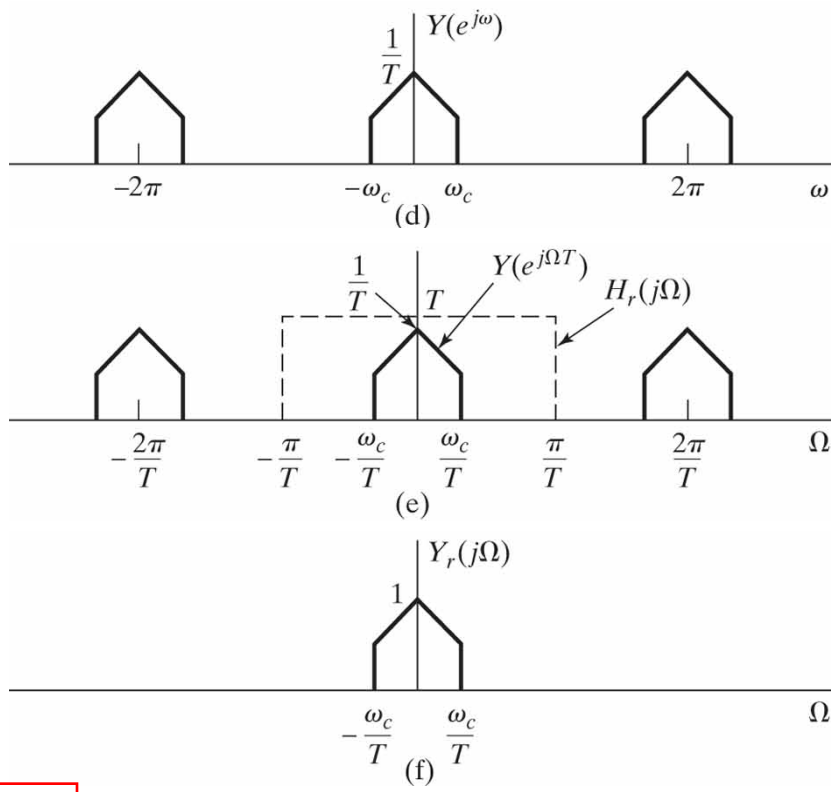
(c) F.T. $X(e^{j\omega})$ of sequence of samples and frequency response $H(e^{j\omega})$ of D.T. system plotted versus ω.



(d) F.T. of output of D.T. system.

(e) F.T. of output of D.T. system and frequency response of ideal reconstruction filter plotted versus Ω.

(f) F.T. of output.

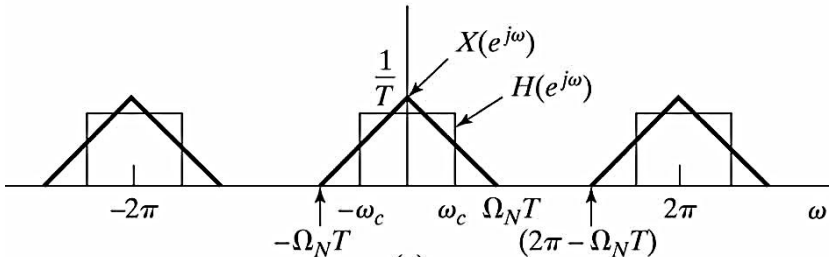


No aliasing: $\boxed{(2\pi - \Omega_N T) \geq \omega_c}$

Example 3 (cont'd)

- **The effective filter below still holds even if aliasing occurs, provided aliased or distorted components are eliminated by D.T. filter $H(e^{j\omega})$**

$$H_{eff}(j\Omega) = \begin{cases} 1 & |\Omega T| < \omega_c \text{ or } |\Omega| < \omega_c/\mathrm{T} \\ 0 & |\Omega T| \geq \omega_c \text{ or } |\Omega| \geq \omega_c/\mathrm{T} \end{cases}$$



No aliasing: $(2\pi - \Omega_N T) \geq \omega_c$

Nyquist condition: $(2\pi - \Omega_N T) \geq \Omega_N T$

Example 4: D.T. Implementation of an Ideal C.T. Differentiator

- **Ideal C.T. differentiator is defined by**

$$y_c(t) = \frac{d}{dt}\left[x_c(t)\right] \qquad H_c(j\Omega) = j\Omega$$

- **We want to realize this filter using a C/D converter, D.T. filter, D/C converter**
  - Inputs are band-limited
  - Equivalent effective C.T. filter becomes

$$H_{eff}(j\Omega) = \begin{cases} j\Omega, & |\Omega| < \pi/\mathrm{T} \\ 0, & |\Omega| \geq \pi/\mathrm{T} \end{cases} = \begin{cases} H\left(e^{j\Omega}\right), & |\Omega| < \pi/T \\ 0, & |\Omega| \geq \pi/T \end{cases}$$

- **D.T. filter then become** $H\left(e^{j\omega}\right) = \frac{j\omega}{T}, \quad |\omega| < \pi$

- **Impulse response:**

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{j\omega}{T}\right)e^{j\omega n}d\omega$$

$$= \frac{\pi n\cos\pi n - \sin\pi n}{\pi n^2 T}$$

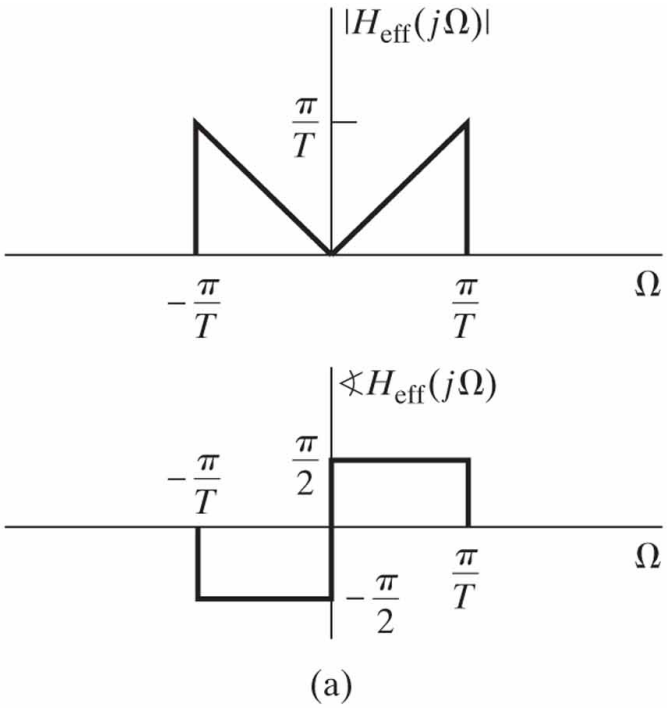
$$= \begin{cases} 0 & n = 0 \\ \dfrac{\cos\pi n}{nT} & n \neq 0 \end{cases}$$

Apply 'Hôpital's rule

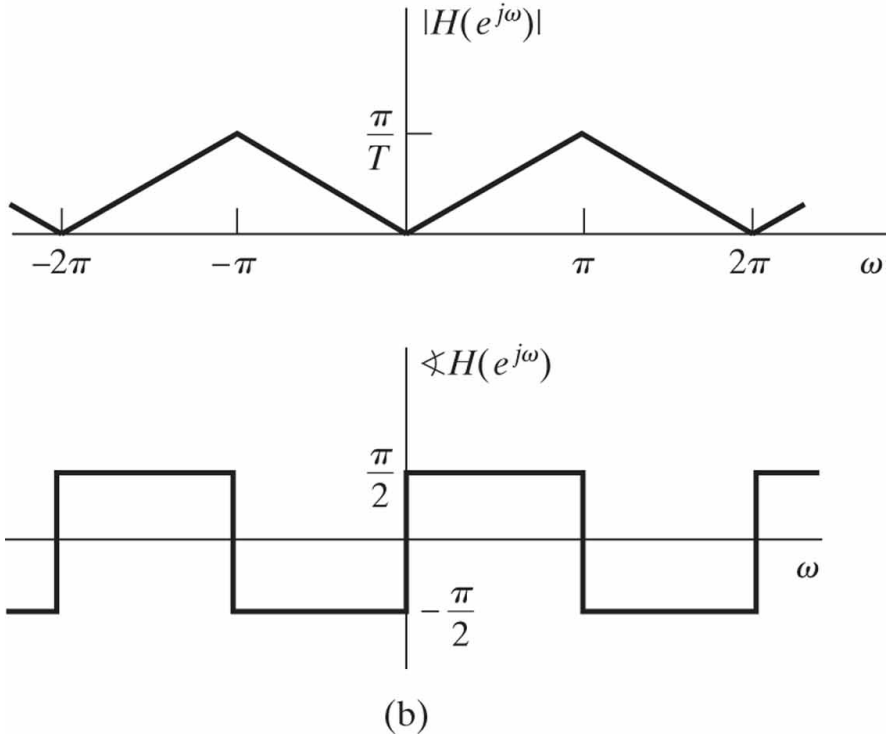
Example 4: D.T. Implementation of an Ideal C.T. Differentiator

$$H_{eff}(j\Omega) = \begin{cases} j\Omega, & |\Omega| < \pi/\mathrm{T} \\ 0, & |\Omega| \geq \pi/\mathrm{T} \end{cases}$$

$$H\left(e^{j\omega}\right) = \frac{j\omega}{T}, \quad |\omega| < \pi$$



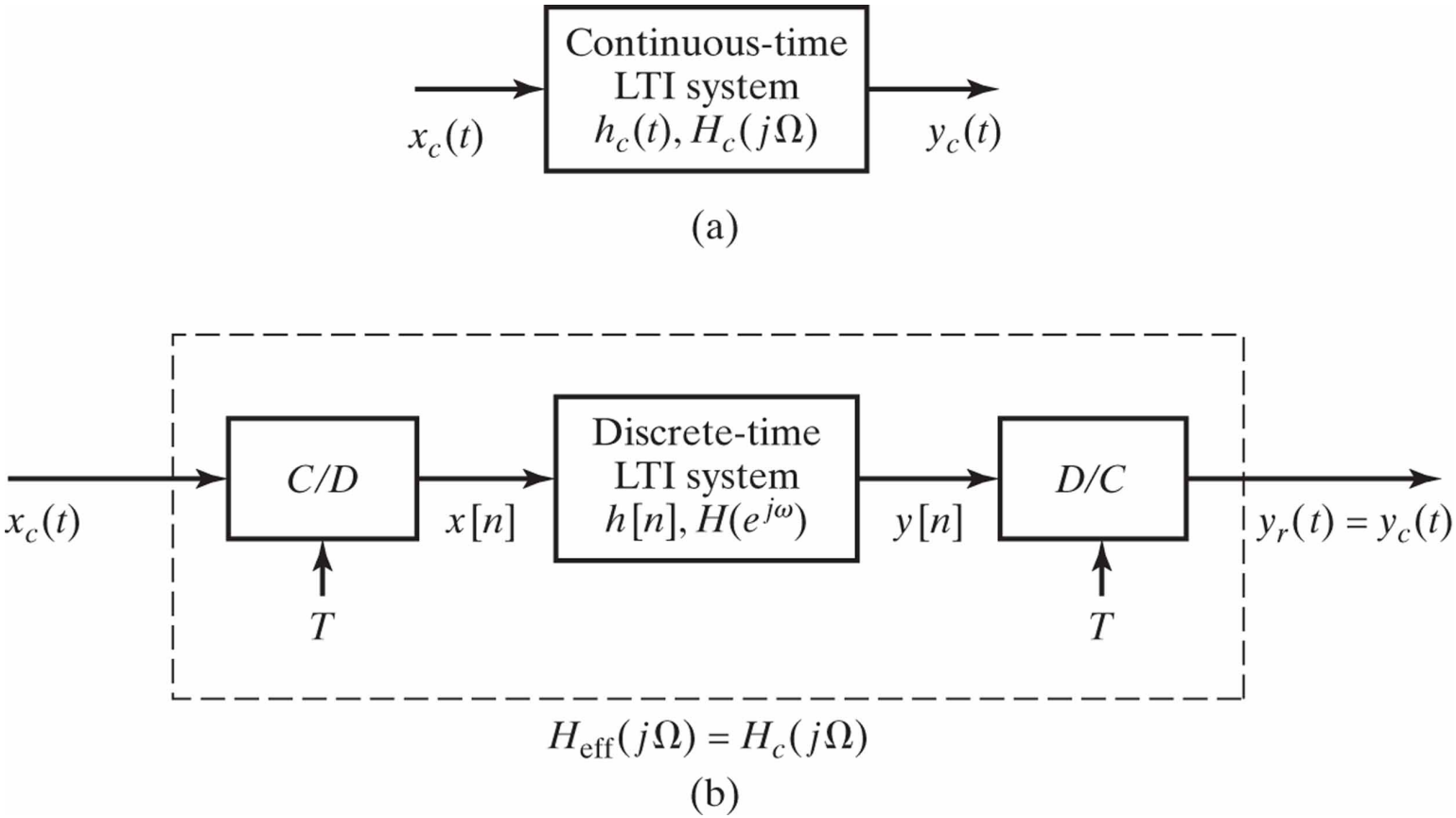
(a)



(b)

Impulse Invariance (Revisited)

- **Given a desired C.T. we wish to implement using a D.T. system as follows:**



Impulse Invariance (Revisited)

- **With $H_c(j\ \ )$ band-limited, we choose $H(e^{j}\ \ )$ so that:**

$$H_{eff}(j\Omega) = H_c(j\Omega)$$

$$\Rightarrow H\left(e^{j\omega}\right) = H_c(j\omega/T), \quad |\omega| < \pi$$

Also $\quad H_c(j\Omega) = 0, \quad |\Omega| \geq \pi/T$

- **What is resulting relationship between $h_c(t)$ and $h[n]$?**
  - $h[n]$ is a scaled, sampled version of $h_c(t)$, under the impulse invariance condition

$$\text{if} \quad h[n] = h_c(nT)$$

$$\text{then} \quad H\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right)$$

$$\text{But:} \quad H_c(j\Omega) = 0, \quad |\Omega| \geq \pi/T$$

$$\text{Therefore,} \quad H\left(e^{j\omega}\right) = \frac{1}{T}H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

$$\text{So to get } H\left(e^{j\omega}\right) = H_c\left(j\frac{\omega}{T}\right) \text{ for } |\omega| < \pi, \text{ we scale } h[n] \text{ so that } h[n] = T \cdot h_c(nT)$$

Example 5: D.T. LPF Obtained by Impulse Invariance

- Design an ideal D.T. LPF with cutoff frequency $\omega_c < \pi$ by sampling a C.T. LPF with cutoff frequency Ω_c.

- **Solution**: We have $$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & |\Omega| \geq \Omega_c \end{cases}$$

- The cutoff frequency ω_c is related to Ω_c by
$$\omega_c = \Omega_c T$$

- Impulse response of C.T. LPF is
$$h_c(t) = \frac{\sin \Omega_c t}{\pi t}$$

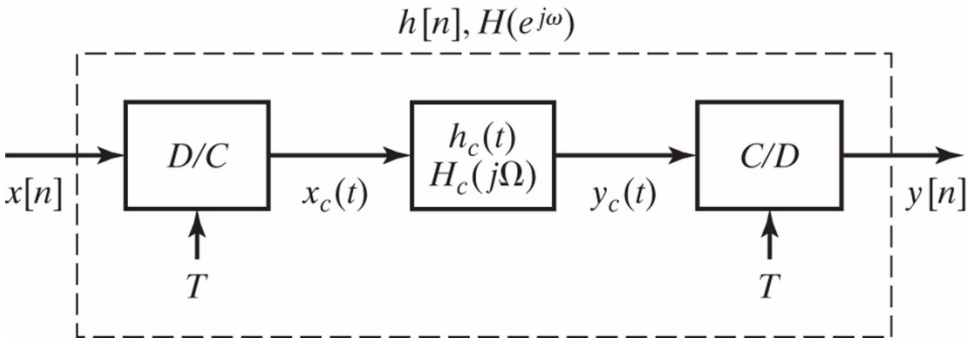
- Using impulse invariance, we obtain

$$\begin{aligned} h[n] &= T \cdot h_c(nT) \\ &= T \frac{\sin(\Omega_c nT)}{\pi nT} \\ &= \frac{\sin(\omega_c n)}{\pi n} \end{aligned}$$

$$\Rightarrow \quad H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

C.T. Processing of D.T. Signals

C.T. Processing of D.T. Signals



- **From definitions of D/C and C/D, we must have $X_c(j\Omega)=0$, $Y_c(j\Omega)=0$ for $|\Omega| \geq \pi/T$**

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

- **In frequency domain:**

$$X_c(j\Omega) = TX\left(e^{j\Omega T}\right), \quad |\Omega| < \pi/T$$

$$Y_c(j\Omega) = H_c(j\Omega) X_c(j\Omega)$$

$$Y\left(e^{j\omega}\right) = \frac{1}{T} Y_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

$$\Rightarrow H\left(e^{j\omega}\right) = H_c\left(j\frac{\omega}{T}\right), \quad |\omega| < \pi$$

$$\text{or } H_c(j\Omega) = H\left(e^{j\Omega T}\right), \quad |\Omega| < \pi/T$$

Example 6: Non-Integer Delay

- **Consider a D.T. system with frequency response:**

$$H\left(e^{j\omega}\right) = e^{-j\omega\Delta}, \quad |\omega| < \pi$$

$$y[n] = x[n-\Delta], \quad \Delta \text{ not necessarily an integer}$$

- **We seek an interpretation in the C.T. domain.**
- **Choose a C.T. filter with response:**

$$H_c(j\Omega) = H\left(e^{j\Omega T}\right) = e^{-j\Omega T\Delta}$$

$$y_c(t) = x_c(t - T\Delta)$$

  - Here, $x_c(t)$ is band-limited interpolation of $x[n]$; $y[n]$ is obtained by sampling $y_c(t)$
- **Example: if Δ=1/2, *y*[*n*] would be the band-limited interpolation halfway between the input sequence values.**

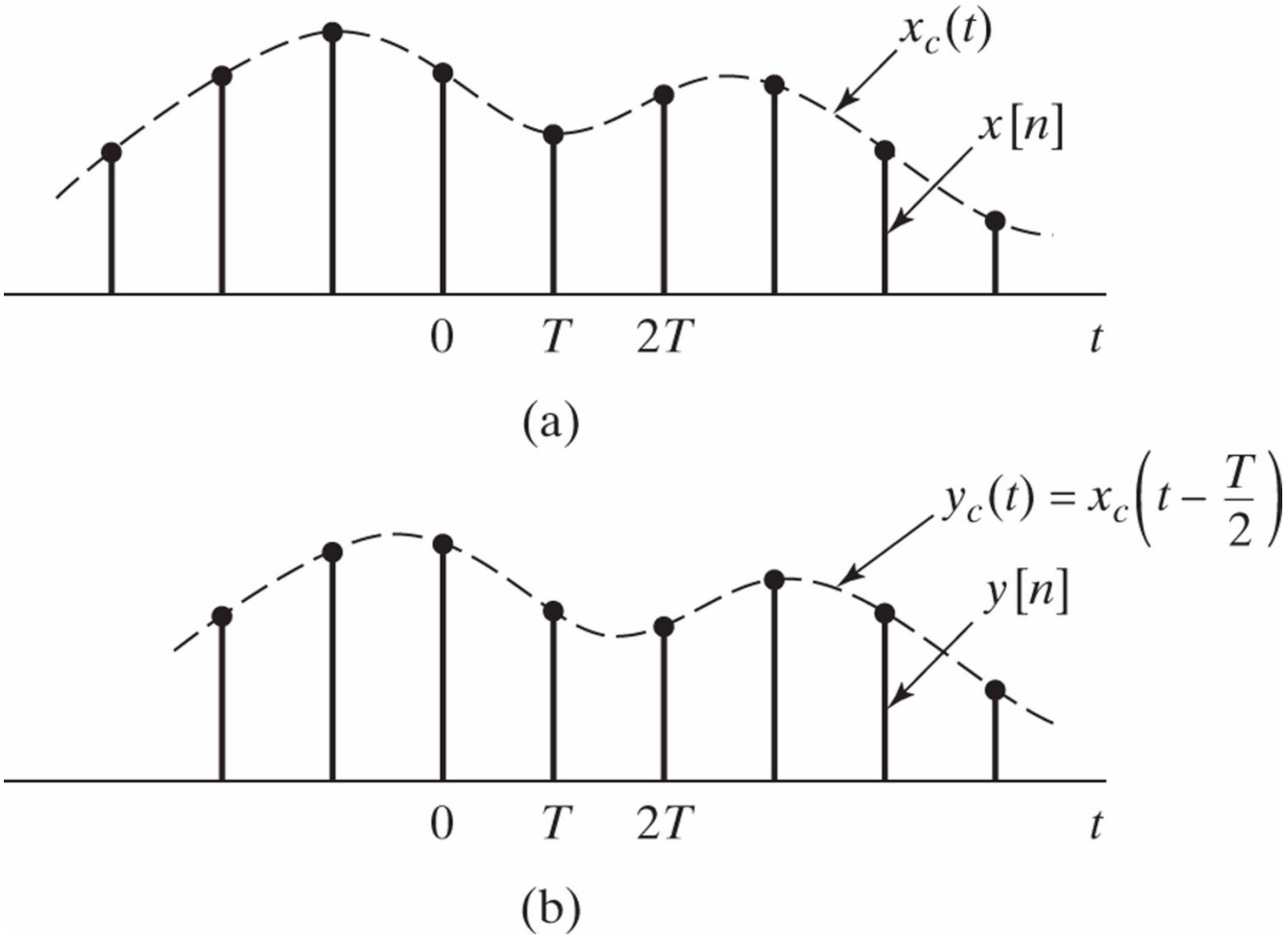
$$y[n] = y_c(nT) = x_c(nT - T\Delta)$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left[\pi(t - T\Delta - kT)/T\right]}{\pi(t - T\Delta - kT)/T}\Bigg|_{t=nT}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \frac{\sin\left[\pi(n-k-\Delta)\right]}{\pi(n-k-\Delta)} \longrightarrow \equiv x[n] * \frac{\sin\left[\pi(n-\Delta)\right]}{\pi(n-\Delta)}$$

Example 6: Non-Integer Delay

(a) C.T. processing of the D.T. sequence (b) can produce a new sequence with a "half-sample" delay.

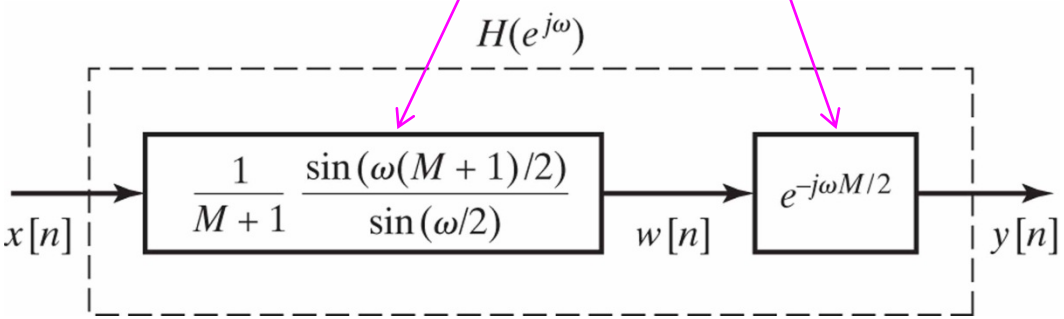


Example 7: Moving-Average System with Non-Integer Delay

- **For an (*M*+1)-point causal moving average system, we have**

$$H\left(e^{j\omega}\right)=\frac{1}{M+1}\frac{\sin\left[\omega(M+1)/2\right]}{\sin(\omega/2)}e^{-j\omega M/2}, \quad |\omega|<\pi$$

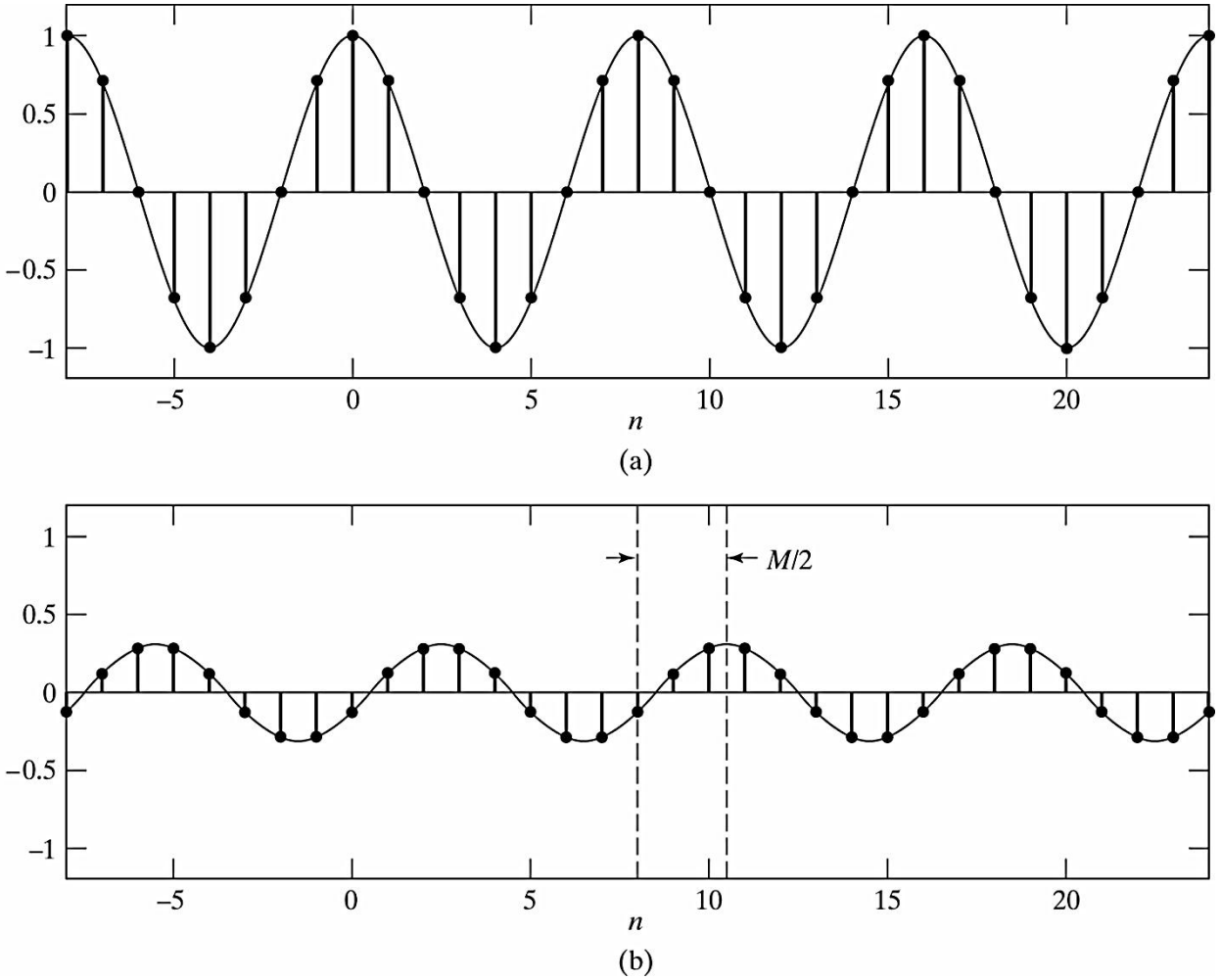
- **Can represent as a cascade of two systems:**



- **If *M* is even, then linear phase term is an integer delay**
- **If *M* is odd, then linear phase term is a non-integer delay (integer-plus-one-half)**
  - *y*[*n*] is then the band-limited interpolation of *w*[*n*]
  - Followed by a C.T. delay of *MT*/2, where *T* is sampling period of D/C interpolation of *w*[*n*])
  - Followed by C/D conversion with sampling period *T*.

Example 7 (cont'd)

Illustration of moving-average filtering. (a) Input signal $x[n] = \cos(0.25\pi n)$. (b) Corresponding output of six-point moving-average filter.



$M = 5$

Example 7 (cont'd)

$$x[n] = \cos(0.25\pi n) = \frac{1}{2}e^{j0.25\pi n} + \frac{1}{2}e^{-j0.25\pi n}$$

$$H\left(e^{j\omega}\right) = \frac{1}{M+1}\frac{\sin\left[\omega(M+1)/2\right]}{\sin(\omega/2)}e^{-j\omega M/2}, \quad |\omega| < \pi$$

$$y[n] = H\left(e^{j0.25\pi}\right)\frac{1}{2}e^{j0.25\pi n} + H\left(e^{-j0.25\pi}\right)\frac{1}{2}e^{-j0.25\pi n}$$
$$= 0.308\cos\left[0.25\pi(n-2.5)\right]$$

Moving average filter:

- Reduces amplitude of input
- Introduces a phase shift corresponding to 2.5 samples of delay