EECE 491: Discrete-time Signal Processing

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Lecture 5: Frequency Domain Analysis of LTI Systems

Announcements

- **Reading**
  - Chapters 4 and 5, Proakis

Outline

- **Part I: Frequency analysis of signals**
- **Part II: Frequency analysis of LTI systems**
- **Part III: LTI Systems as Frequency Selective Filters**

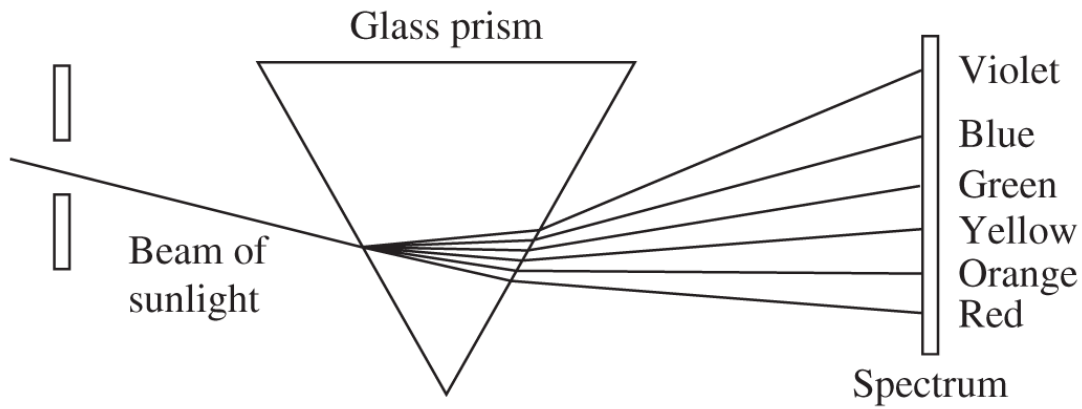
Part I: Frequency analysis of signals

Frequency-Domain and Time-Domain Signal Properties

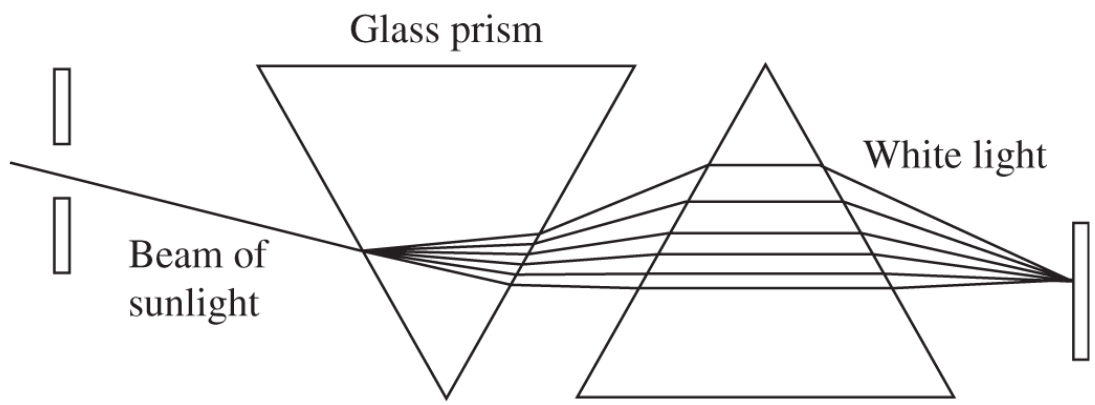
- **Continuous-time**
  - Fourier Series for periodic signals
  - Fourier Transform for aperiodic signals

- **Discrete-time**
  - Fourier Series for periodic signals
  - Fourier Transform for aperiodic signals

Frequency Analysis: Analogy to Analysis of White Light Spectrum

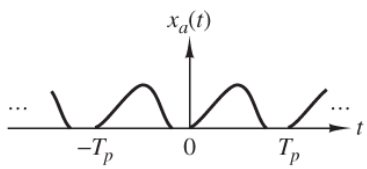
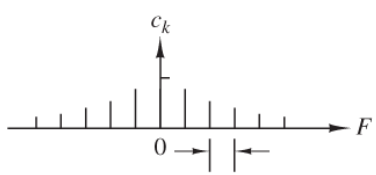
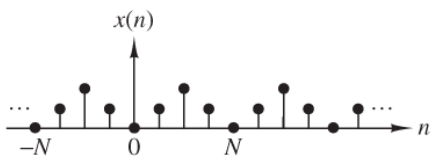
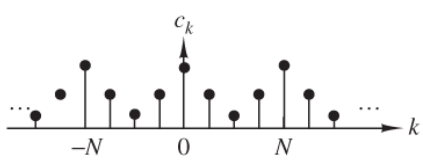
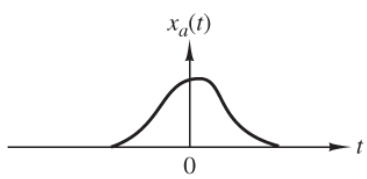
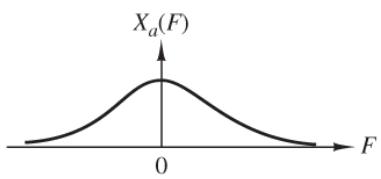
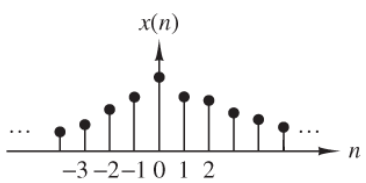
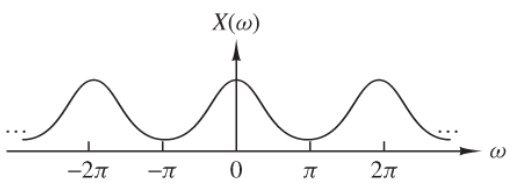


(a)



(b)

Fourier Representations of Signals

		Continuous-time signals		Discrete-time signals	
		Time-domain	Frequency-domain	Time-domain	Frequency-domain
Periodic signals	Fourier series	 $c_k = \frac{1}{T_p}\int_{T_p} x_a(t)e^{-j2\pi kF_0 t}\,dt$	 $F_0 = \frac{1}{T_p}$ $x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$	 $c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)kn}$	 $x(n) = \sum_{k=0}^{N-1} c_k e^{j(2\pi/N)kn}$
		Continuous and periodic	Discrete and aperiodic	Discrete and periodic	Discrete and periodic
Aperiodic signals	Fourier transforms	 $X_a(F) = \int_{-\infty}^{\infty} x_a(t)e^{-j2\pi Ft}\,dt$	 $x_a(t) = \int_{-\infty}^{\infty} X_a(F)e^{j2\pi Ft}\,dF$	 $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$	 $x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\,d\omega$
		Continuous and aperiodic	Continuous and aperiodic	Discrete and aperiodic	Continuous and periodic

I - Fourier Series for Continuous-Time Periodic Signals

- $x(t)$ **continuous-time, finite/periodic on** $[-\pi,\pi]$ **with period** $T_p = 1/F_0$

Power signal

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

$$c_k = \frac{1}{T_p}\int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

Sometimes we use notation

$$c_k \quad \leftrightarrow \quad X(k)$$

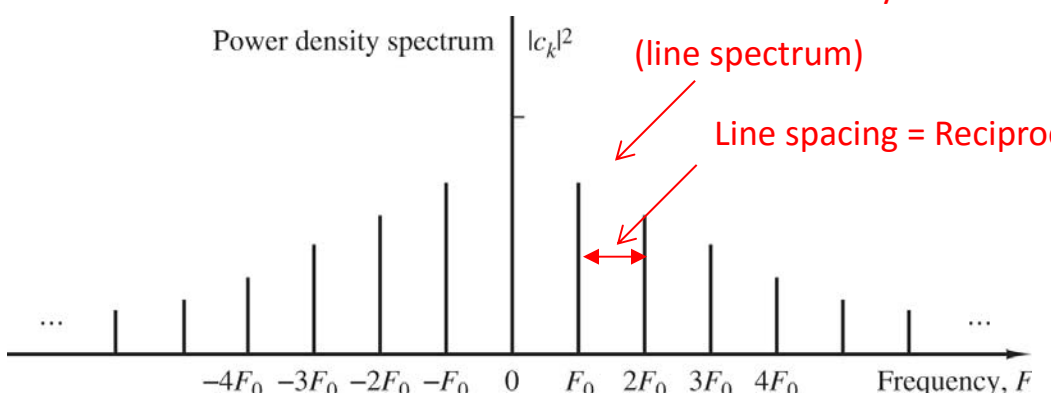
- **Fourier coefficients c_k: complex valued in general**
  - If $x(t)$ even, c_k's are real

- **Power density spectrum:** $P_x = \frac{1}{T_p}\int_{T_p} |x(t)|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2$ **(Parseval's theorem for power signals)**

Power exists only at discrete frequencies

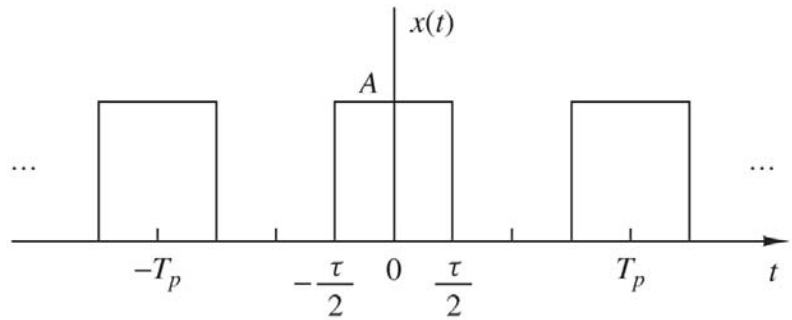
(line spectrum)

Line spacing = Reciprocal of period



Example 1

- **Determine Fourier Series and power density spectrum of**

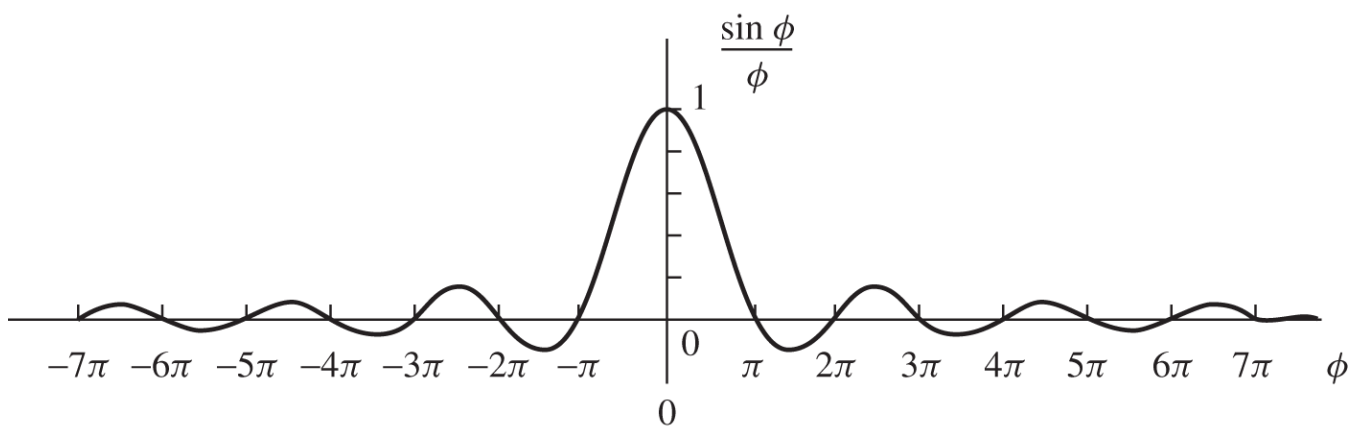


Pulse width τ
Period T_p

$$c_0 = \frac{1}{T_p}\int_{T_p} A\,dt = \frac{A\tau}{T_p}$$

$$c_k = \frac{A\tau}{T_p}\frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}, \quad k = \pm 1, \pm 2, \cdots$$

Samples of this plot at $\phi = \pi k F_0 \tau$

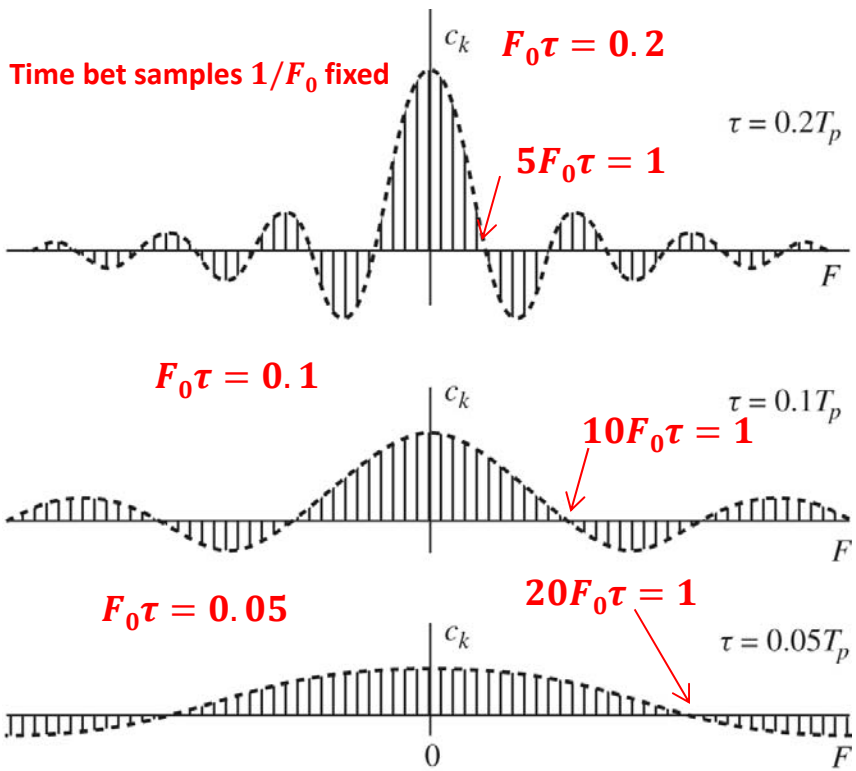


Example 1 (cont'd)

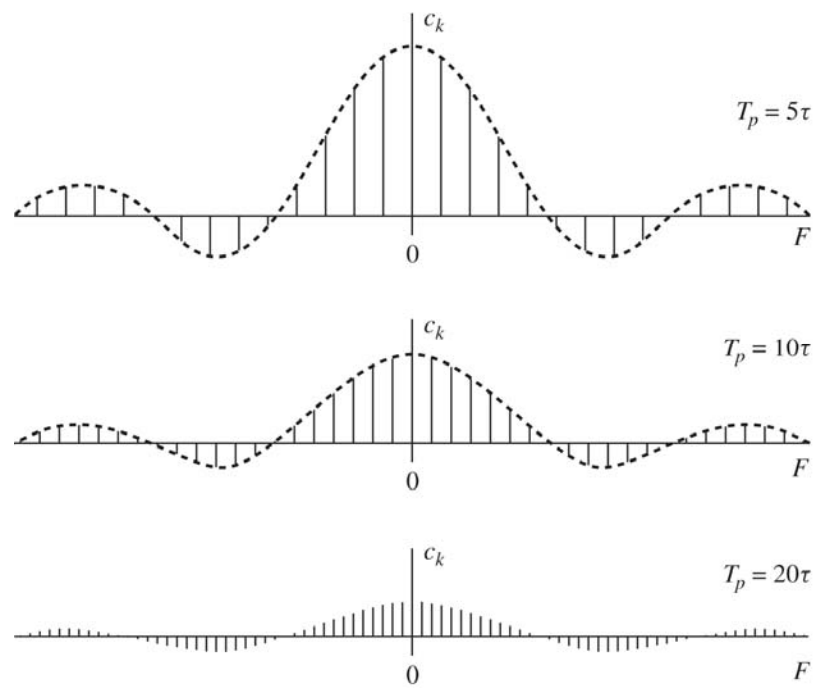
$$|c_0|^2 = \left(\frac{A\tau}{T_p}\right)^2 \qquad |c_k|^2 = \left(\frac{A\tau}{T_p}\right)^2 \left(\frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}\right)^2, \qquad k = \pm 1, \pm 2, \cdots$$

T_p fixed; pulse width τ varies

Time bet samples $1/F_0$ fixed



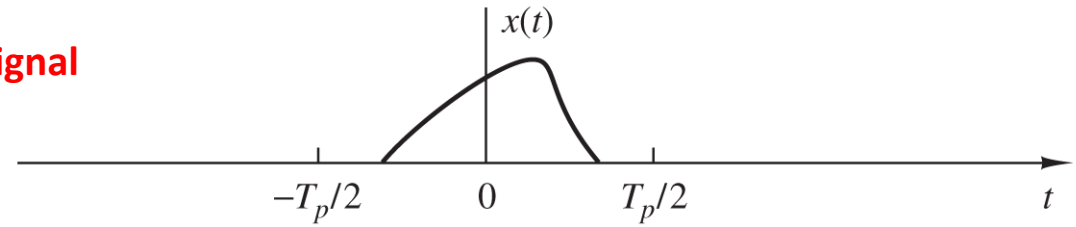
Fixed pulse width τ ; period T_p varies



II - Fourier Transform for Continuous Time Aperiodic Signals

- ***x*(*t*): continuous-time, aperiodic with finite duration on [-T_p/2, +T_p/2]**

Energy signal



$$x(t) \leftrightarrow X(F)$$

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$

complex

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

(spectrum)

$$\Omega = 2\pi F$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega$$

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

- **Energy:**

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF$$

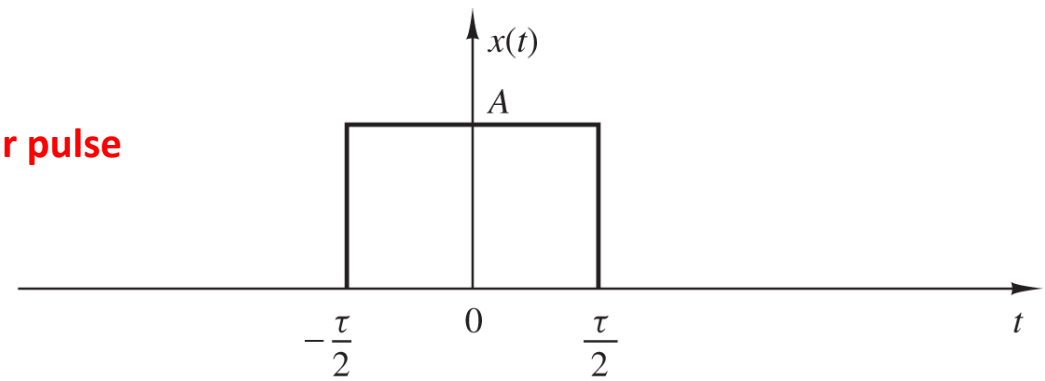
Parseval's theorem for aperiodic finite energy signals

$$S_{xx} = |X(F)|^2$$

Energy density spectrum

Example 2

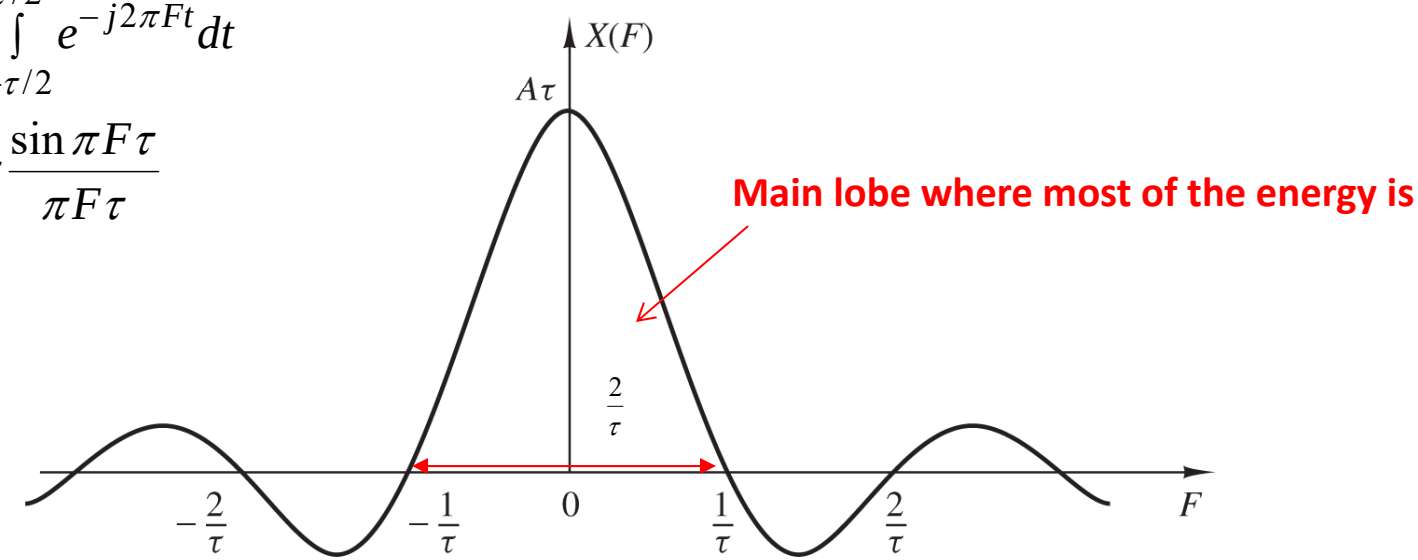
Rectangular pulse



(a)

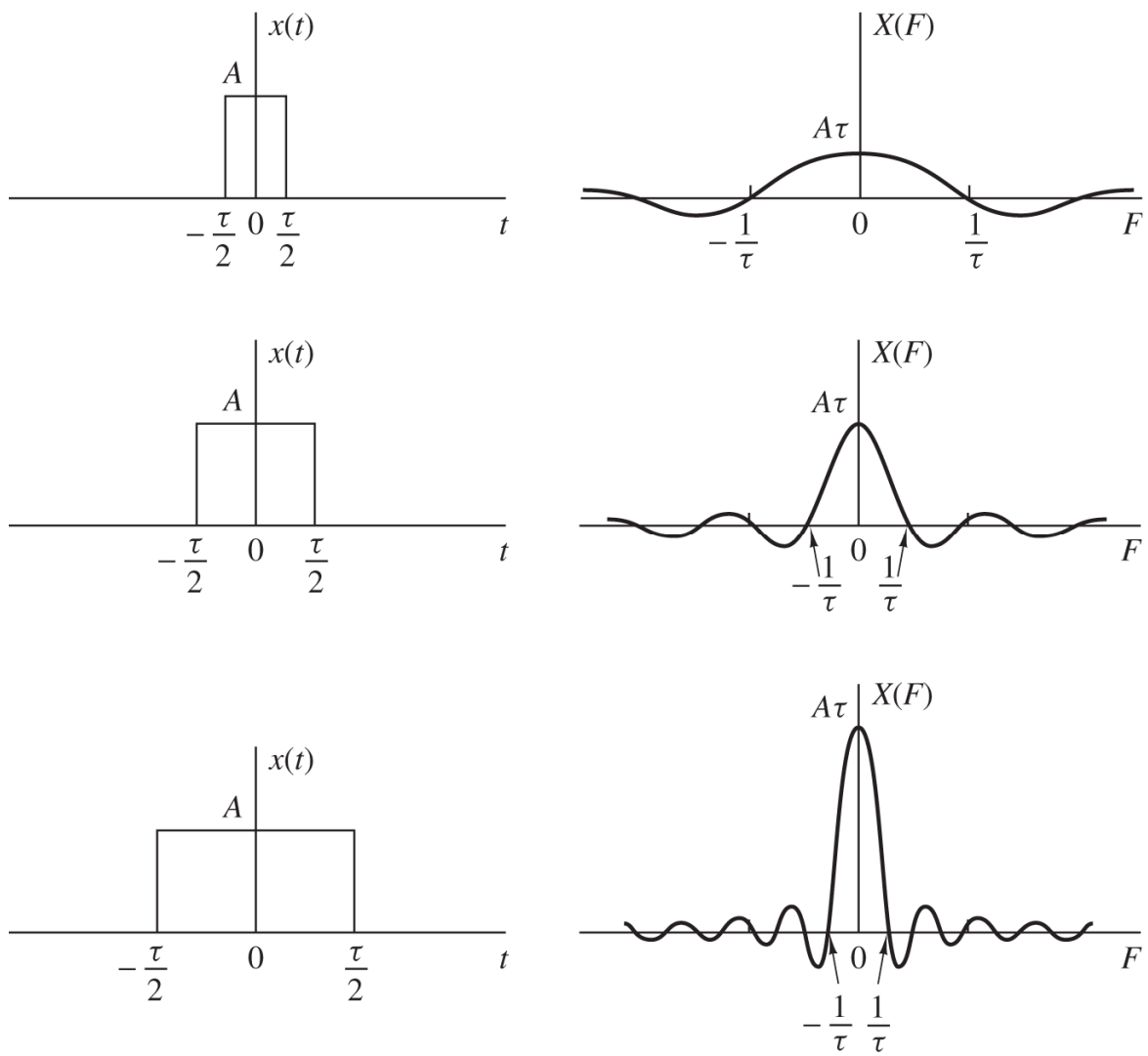
$$X(F) = A\int_{-\tau/2}^{\tau/2} e^{-j2\pi Ft}dt$$

$$= A\tau\frac{\sin \pi F\tau}{\pi F\tau}$$



(b)

Example 2 (cont'd)



III - Fourier Series for Discrete-Time Periodic Signals

- **CT signal: periodic with period T_p**
  - Frequency range extends from -∞ to ∞. Can have infinite number of frequency components
- **DT signal: periodic with period *N***
  - Frequency range is -π to π. Can have at most *N* frequency components separated by $2\pi / N$ radians or $f = 1/N$ cycles

$$\boxed{\begin{aligned} x(n) &= \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \\ c_k &= \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \end{aligned}}$$

Sometimes we use notation

$$c_k \quad \leftrightarrow \quad X(k)$$

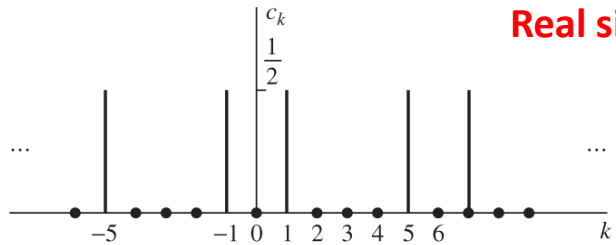
$$c_{k+N} = c_k \Rightarrow \quad \text{periodic}$$

- **Spectrum of a periodic signal *x*(*n*) with period *N* is a periodic sequence as well with a period *N*.**

- **Power density spectrum:** $P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |c_k|^2$

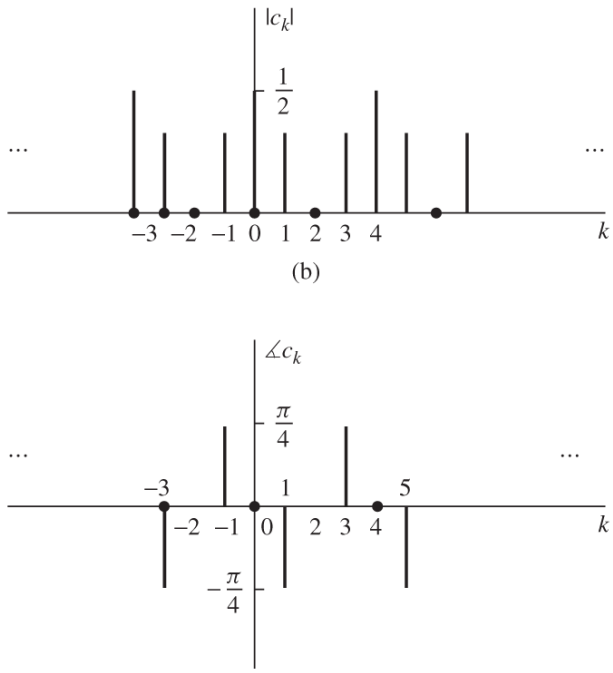
Example 3

Spectrum of $x(n) = \cos\sqrt{2}\pi n$



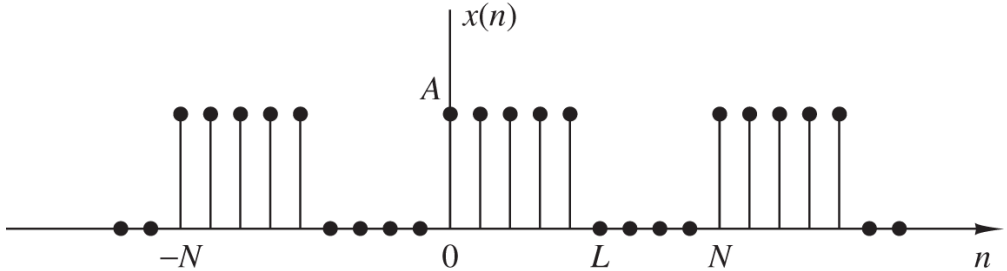
Real since *x*(*n*) even

Spectrum of $x(n) = \{\underline{1},1,0,0\}$
with period 4



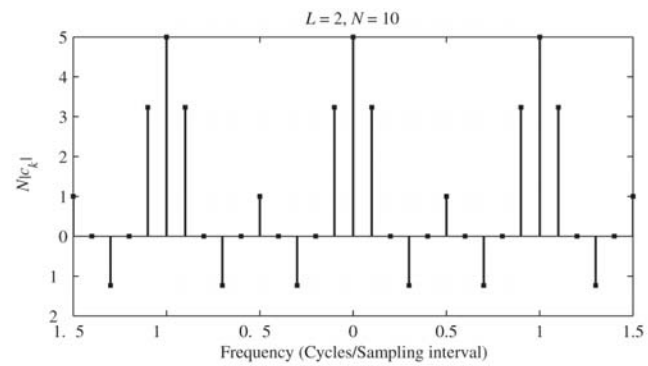
Example 4

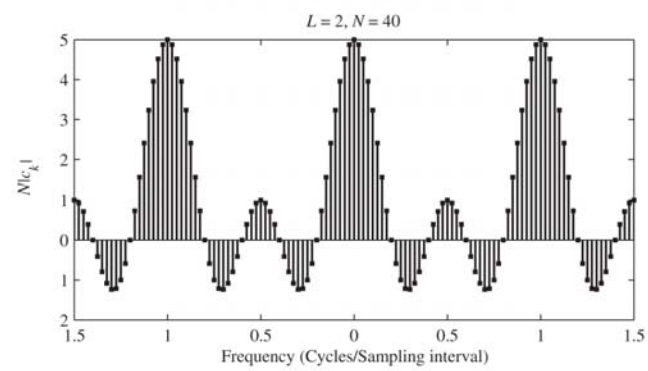
- **Find spectrum of the DT periodic square wave below**



$$c_k = \frac{1}{N}\sum_{n=0}^{L-1} A e^{-j2\pi kn/N}, \quad k = 0,1,\cdots,N-1$$

$$= \begin{cases} \dfrac{AL}{N} & k = 0, \pm N, \cdots \\ \dfrac{A}{N} e^{-j\pi k(L-1)/N} \dfrac{\sin(\pi kL/N)}{\sin(\pi k/N)} & \text{otherwise} \end{cases}$$





IV- Fourier Transform of DT Aperiodic Signals (DTFT)

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Sometimes written as

$$X\left(e^{j\omega}\right)$$

- **Frequency range of $X\left(e^{j\omega}\right)$ is -π to π.**
- $X\left(e^{j\omega}\right)$ **is periodic**
- **Energy:**

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Parseval's theorem for aperiodic finite energy signals

$$S_{xx}(\omega) = |X(\omega)|^2$$

Energy density spectrum

Convergence of the Fourier Transform (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

- **Define the periodic version**

$$X_N(\omega) = \sum_{n=-N}^{N} x(n) e^{-j\omega n}$$

- $X_N(\omega)$ **converges uniformly to** $X(\omega)$ **as** $N \to \infty$
  - That is for every *w*, $X_N(\omega) \to X(\omega)$ as $N \to \infty$
- **Uniform convergence is guaranteed if *x*(*n*) is absolutely summable:**

$$\text{if} \quad \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

$$\Rightarrow \quad |X(\omega)| = \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right| < \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

- **However, some sequences are not absolutely summable, but are square summable, i.e., they have finite energy (weaker condition)**

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

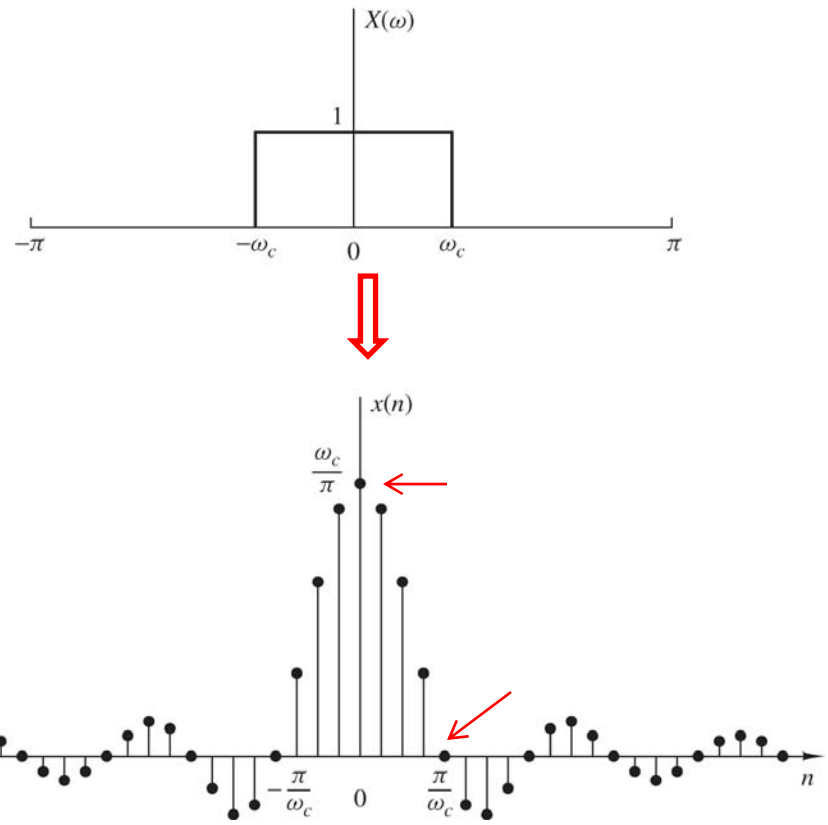
Example 5

- **Consider important class of finite-energy signals:**

$$X(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

- **Its inverse FT is**

$$x(n) = \begin{cases} \dfrac{\omega_c}{\pi} & n = 0 \\ \dfrac{\omega_c}{\pi} \dfrac{\sin \omega_c n}{\pi n} & n \ne 0 \end{cases}$$



Example 5 (cont'd)

- **Now determine the DTFT of**

$$x(n) = \begin{cases} \dfrac{\omega_c}{\pi} & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin \omega_c n}{\pi n} & n \neq 0 \end{cases}$$

- ***x*(*n*) is not absolutely summable but is square summable (i.e. has finite energy). Hence the sequence below converges in the mean-square sense**

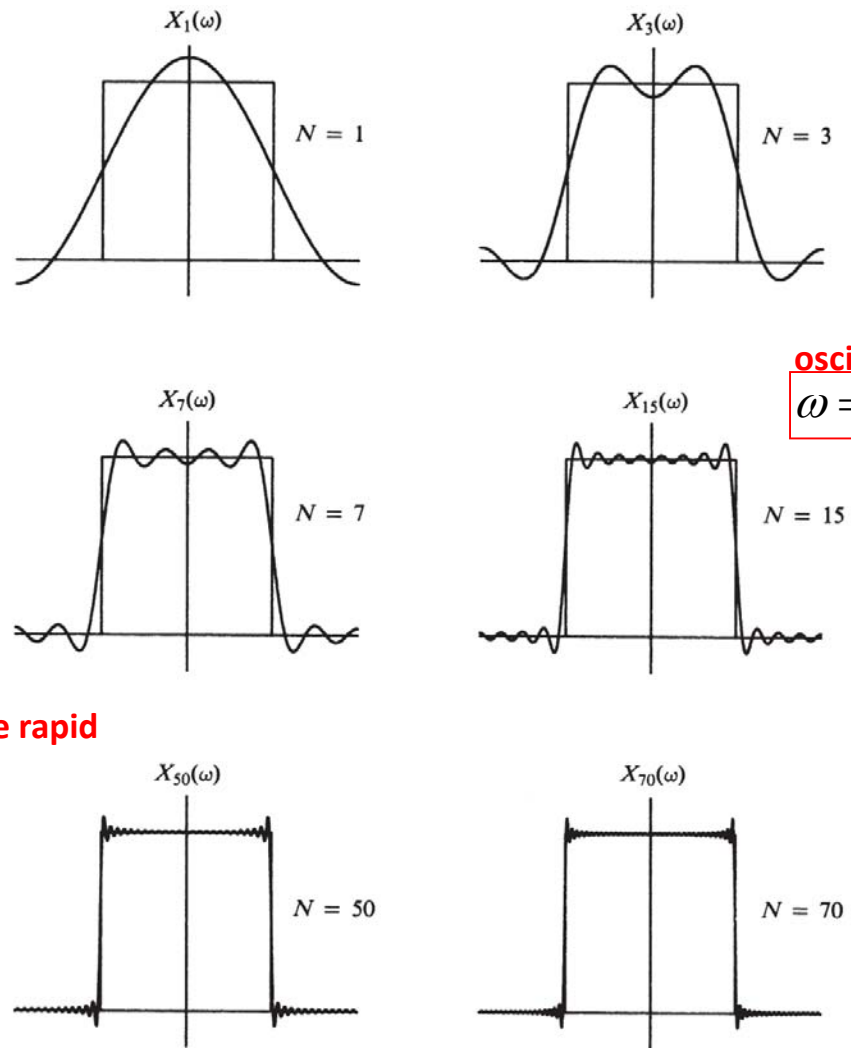
$$\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

- **To illustrate this, define the finite sum**

$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

Example 5 (cont'd): Gibbs Phenomenon

$$X_N(\omega) = \sum_{n=-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$



oscillatory overshoot at $\omega = \omega_c$

As N increases, oscillations become more rapid
But size of ripple stays the same

Summary

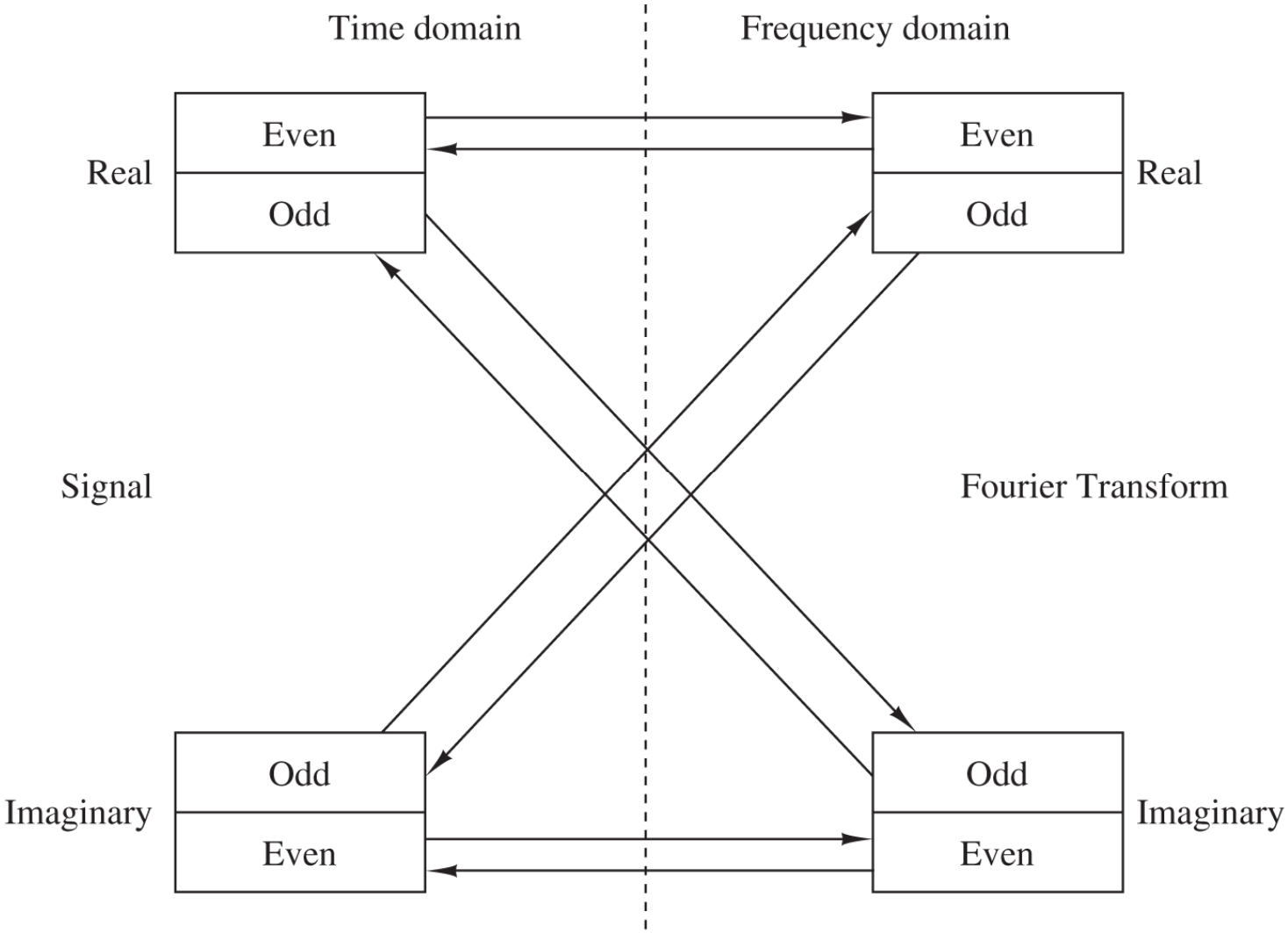
1. **CT signals have aperiodic spectra**
   - Lack of periodicity due to the fact that the complex exponential $e^{j2\pi Ft}$ is a function of a continuous variable t, and hence not periodic in *F*.
   - Frequency range extends from $-\infty$ to ∞
2. **DT signals have periodic spectra: Both the Fourier series and the Fourier transform are periodic with period** $\omega = 2\pi$
3. **Periodic signals have discrete spectra**
   - They are described by means of Fourier series
   - Coefficients provide "lines" that constitute the spectrum
   - Line spacing $\Delta f = 1/N \quad \text{or} \quad \Delta F = 1/T_p$
4. **Aperiodic energy signals have continuous spectra**
   - Follows from the fact that $X(F)$ is a function of $e^{j2\pi Ft}$ and $X(\omega)$ is a function of $e^{j\omega n}$, which are continuous in the variables *F* and ω .

- Periodicity with "period" α in one domain implies discretization with "spacing" $1/\alpha$ in the other domain
- The converse is also true

Symmetry Properties of DTFT

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
$x[n]$ real	*The following properties apply only when $x[n]$ is real:*
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

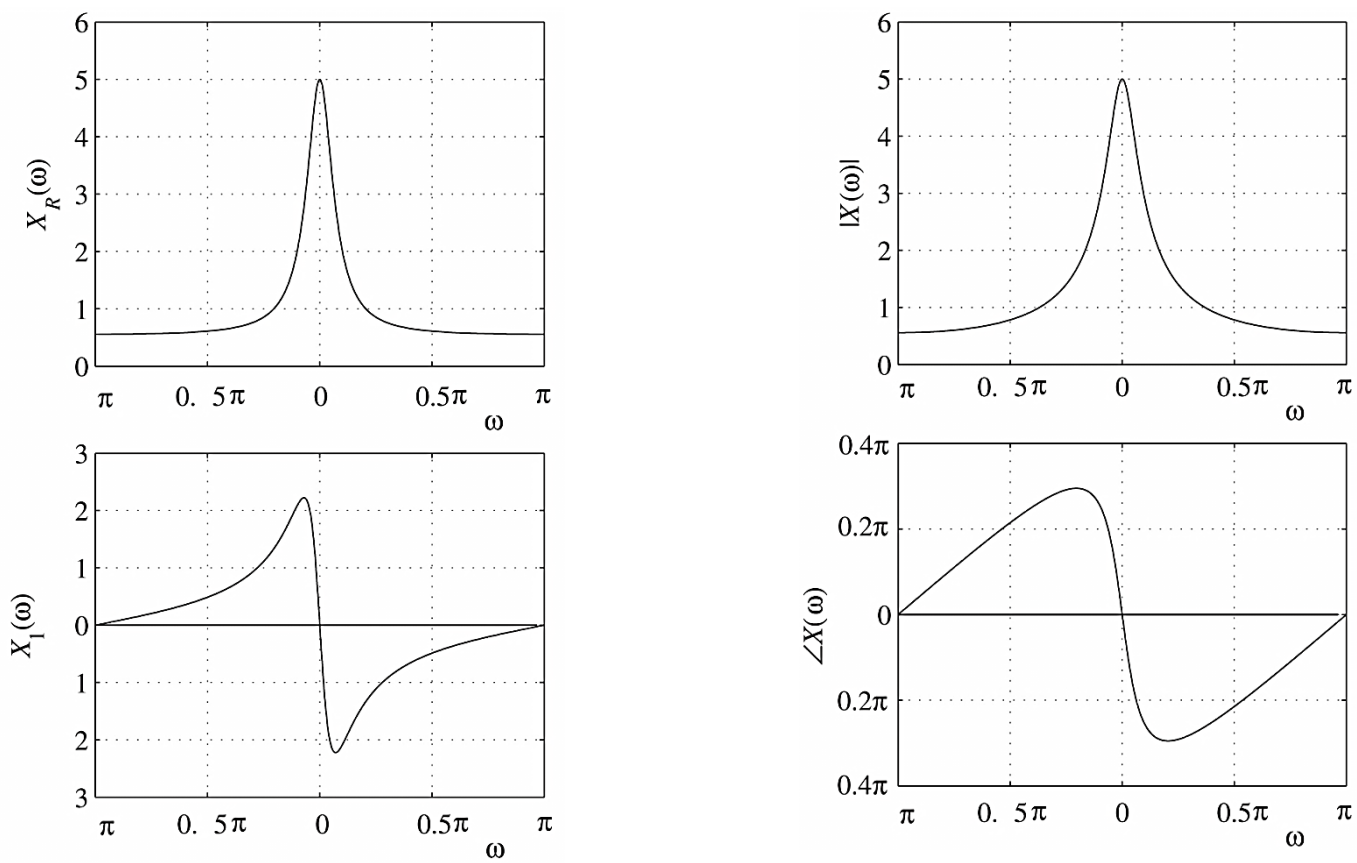
Symmetry Properties of DTFT



Example 6

- **Draw the** $X_R(\omega), X_I(\omega), |X(\omega)|, \measuredangle X(\omega)$ **for the Fourier transform**

$$X(\omega) = \frac{1}{1 - ae^{-j\omega}}, \quad -\pi < \omega < \pi$$



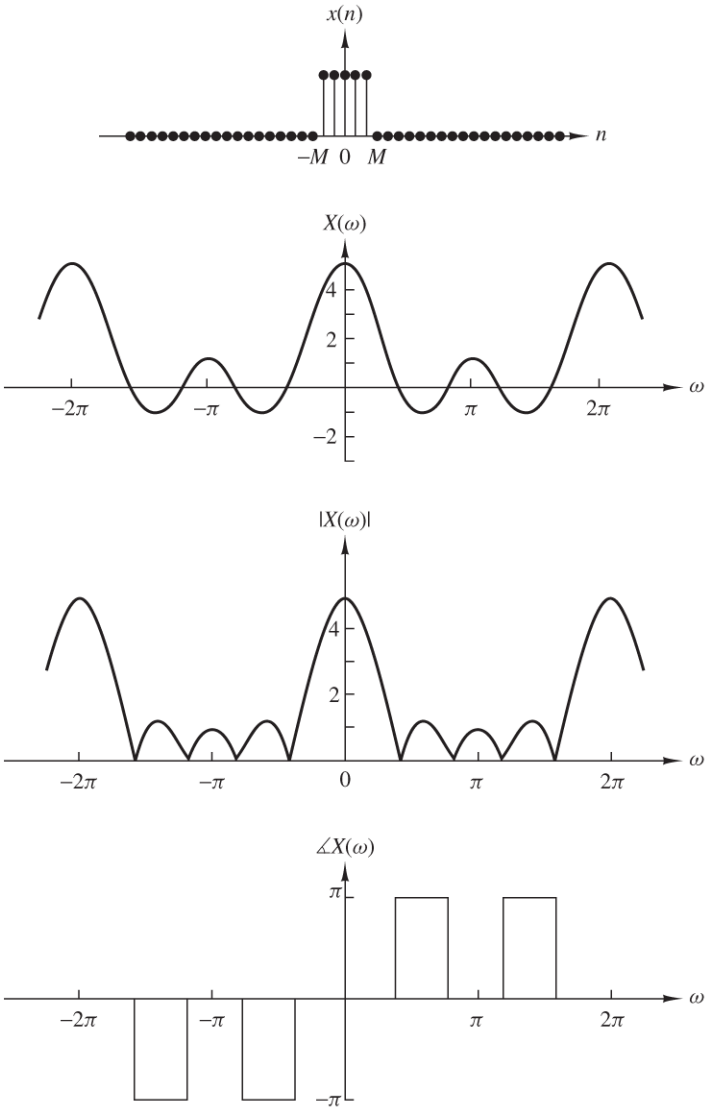
Example 7

- **Determine the DTFT of**

$$x(n) = \begin{cases} A & -M \le n \le M \\ 0 & \text{elsewhere} \end{cases}$$

- **Since *x*(-*n*) = *x*(*n*) then *X*(*w*) is real. Obtain**

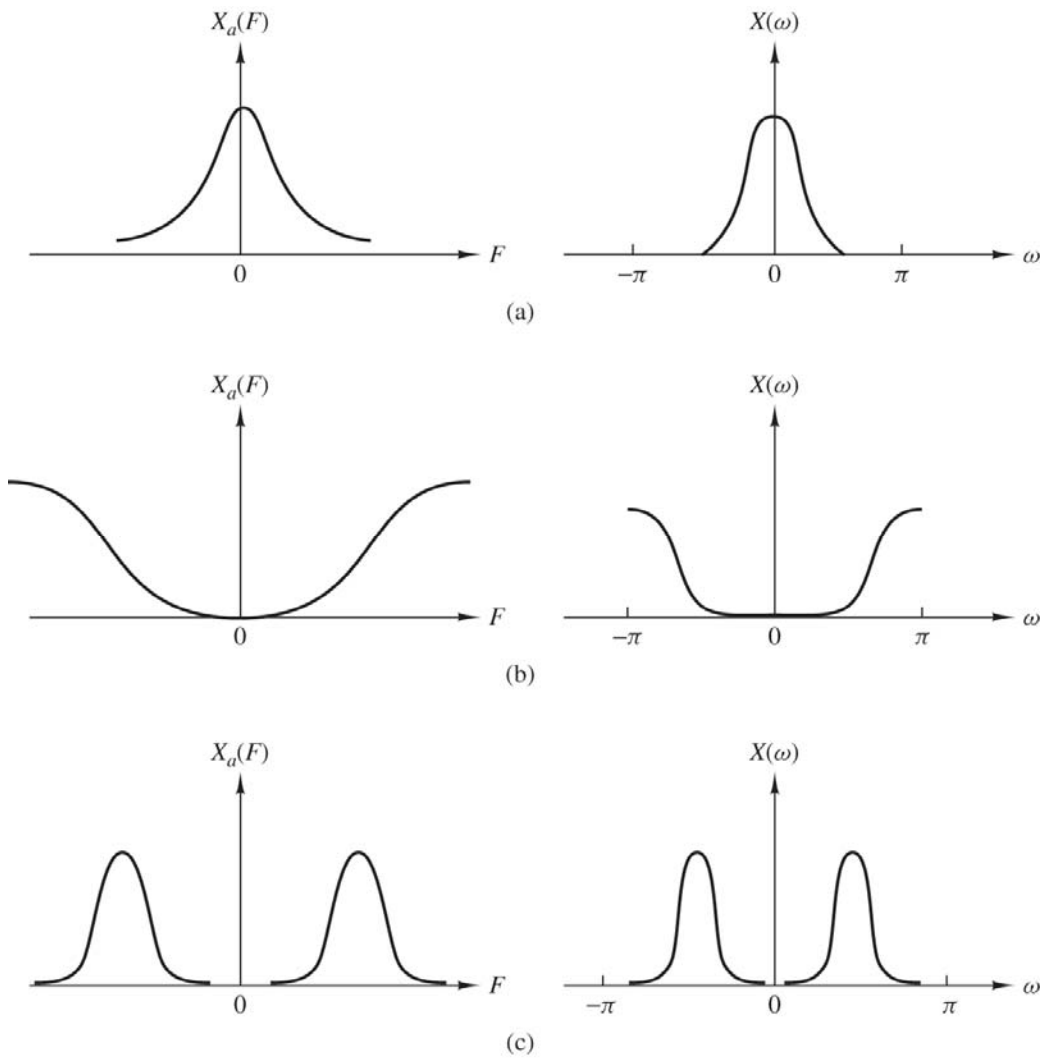
$$X(\omega) = A\frac{\sin\left(M + \frac{1}{2}\right)\omega}{\sin(\omega/2)}$$



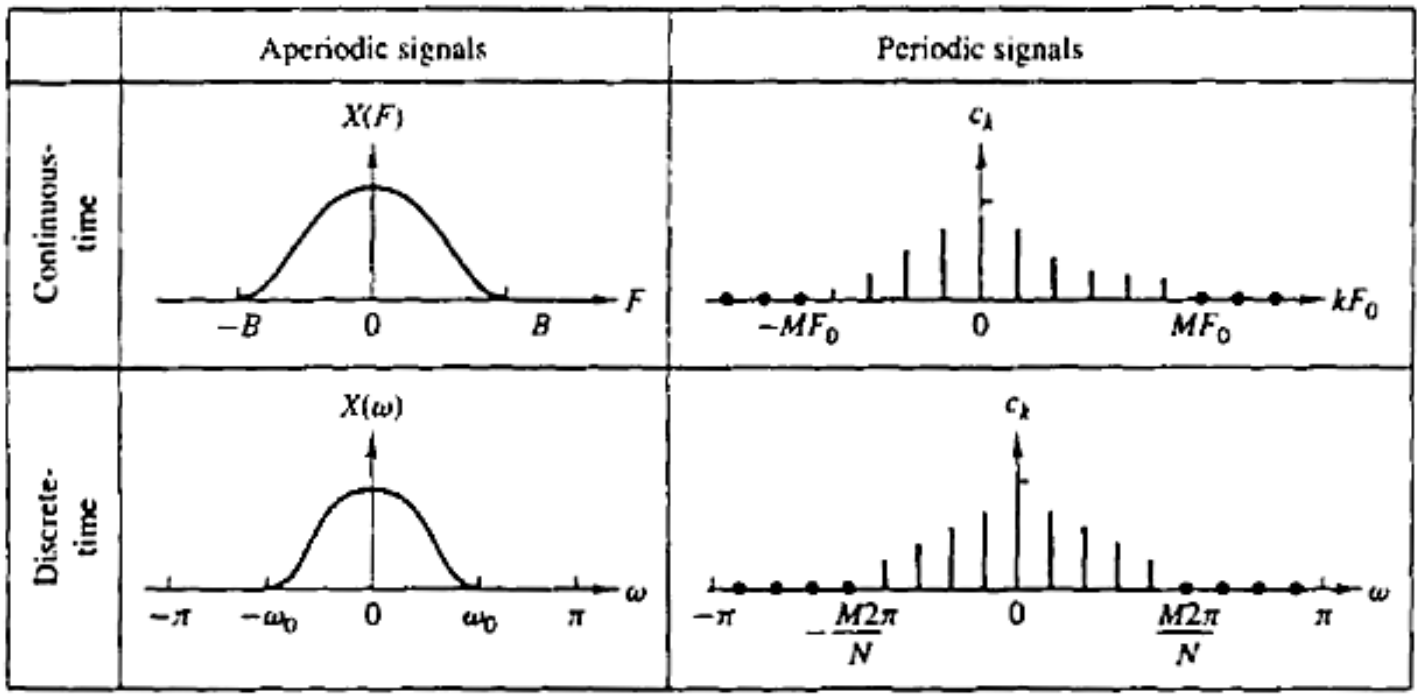
Important Fourier Transform Theorems

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ (n_d an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Parseval's theorem:	
8. $\displaystyle\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	
9. $\displaystyle\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	

The Concept of Bandwidth



Examples of Bandlimited Signals



Outline

Part II: Frequency analysis of LTI systems

Response of LTI Systems to Complex Exponentials

- **LTI system:**

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

- **Excite with an exponential of amplitude *A* and frequency *w*:**

$$x(n) = Ae^{j\omega n}, \qquad -\infty < n < \infty$$

- **Response:**

$$y(n) = A\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]e^{j\omega n}$$

- **The function** $H(\omega)$

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

exists if system is BIBO stable (*h*(*n*) absolutely summable):

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

- **Hence the response is**

$$y(n) = AH(\omega)e^{j\omega n}$$

  – Same frequency, but amplitude multiplied by *H*(*w*)

Example 1

- **Find response of system *h*(*n*) to *x*(*n*):**

$$h(n) = \left(\frac{1}{2}\right)^n u(n) \qquad\qquad x(n) = Ae^{j\pi n/2}, \qquad -\infty < n < \infty$$

- **<u>Solution</u>:**

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

- **Input has frequency** $\omega = \pi / 2$

$$H\left(\frac{\pi}{2}\right) = \frac{1}{1 + j\frac{1}{2}} = \frac{2}{\sqrt{5}} e^{-j26.6^\circ}$$

$$y(n) = A\left(\frac{2}{\sqrt{5}} e^{-j26.6^\circ}\right) e^{j\pi n/2}$$

$$= \frac{2}{\sqrt{5}} A e^{j\left(\pi n/2 - 26.6^\circ\right)}, \quad -\infty < n < \infty$$

Example 2

- **Determine *H*(*w*) for the moving average filter:**

$$y(n)=\frac{1}{3}\left[x(n+1)+x(n)+x(n-1)\right]$$

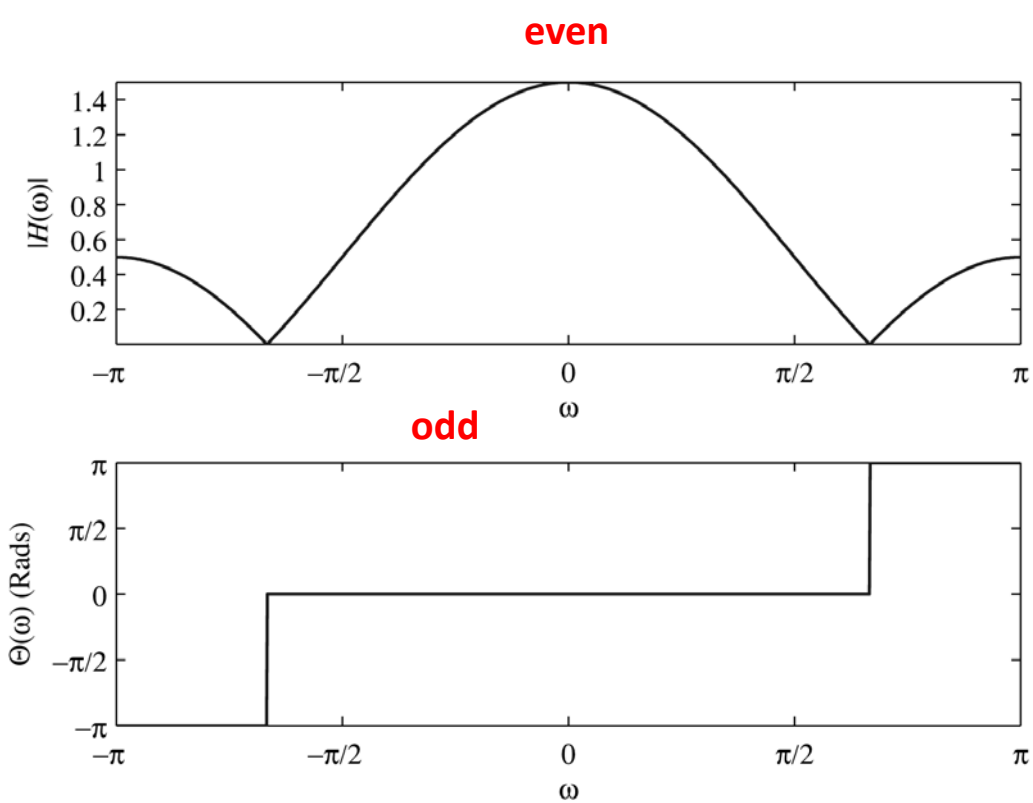
- **<u>Solution</u>:**

$$h(n)=\left\{\frac{1}{3},\underset{\downarrow}{\frac{1}{3}},\frac{1}{3}\right\}$$

$$H(\omega)=\frac{1}{3}\left(e^{j\omega}+1+e^{-j\omega}\right)=\frac{1}{3}(1+2\cos\omega)$$

$$|H(\omega)|=\frac{1}{3}|1+2\cos\omega|$$

$$|\Theta(\omega)|=\begin{cases}0, & 0\le\omega\le 2\pi/3\\ \pi, & 2\pi/3\le\omega<\pi\end{cases}$$



Response to other Sinusoids

$$x_1(n) = Ae^{j\omega n} \quad \rightarrow \quad y_1(n) = A|H(\omega)|e^{j\Theta(\omega)}e^{j\omega n}$$

$$x_2(n) = Ae^{-j\omega n} \quad \rightarrow \quad y_2(n) = A|H(-\omega)|e^{j\Theta(-\omega)}e^{-j\omega n}$$

$$= A|H(\omega)|e^{-j\Theta(\omega)}e^{-j\omega n}$$

$$x(n) = \frac{1}{2}\left[x_1(n) + x_2(n)\right] = A\cos\omega n \rightarrow$$

$$y(n) = \frac{1}{2}\left[y_1(n) + y_2(n)\right] = A|H(\omega)|\cos\left[\omega n + \Theta(\omega)\right]$$

$$x(n) = \frac{1}{j2}\left[x_1(n) - x_2(n)\right] = A\sin\omega n \rightarrow$$

$$y(n) = \frac{1}{2}\left[y_1(n) - y_2(n)\right] = A|H(\omega)|\sin\left[\omega n + \Theta(\omega)\right]$$

Example 3

- **Find response to this input:**

$$x(n) = 10 - 5\sin\frac{\pi}{2}n + 20\cos\pi n, \qquad -\infty < n < \infty$$

$$H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

- **<u>Solution</u>:**

$$@\omega = 0: \qquad H(0) = \frac{1}{1 - \frac{1}{2}} = 2$$

$$@\omega = \frac{\pi}{2}: \qquad H\left(\frac{\pi}{2}\right) = \frac{2}{\sqrt{5}}e^{-j26.6^\circ}$$

$$@\omega = \pi: \qquad H(\pi) = \frac{2}{3}$$

$$y(n) = 20 - \frac{10}{\sqrt{5}}\sin\left(\frac{\pi}{2}n - 26.6^\circ\right) + \frac{40}{3}\cos\pi n, \qquad -\infty < n < \infty$$

Response to Aperiodic Signals

- **From convolution theorem:**

$$Y(\omega) = H(\omega)X(\omega)$$
$$|Y(\omega)| = |H(\omega)||X(\omega)|$$
$$\measuredangle Y(\omega) = \measuredangle H(\omega) + \measuredangle X(\omega)$$

- **Output of an LTI system cannot contain frequency components that are not contained in the input signal**
  - An LTI system can only cause magnitude and phase distortion
- **The output response can be obtained as**

$$y(n) = \frac{1}{2\pi}\int_{2\pi} Y(\omega)e^{j\omega n}d\omega$$

- **Also**

$$|Y(\omega)|^2 = |H(\omega)|^2|X(\omega)|^2$$
$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega)$$

$$\Rightarrow$$

Energy density spectrum

$$E_y = \frac{1}{2\pi}\int_{2\pi} S_{yy}(\omega)d\omega$$
$$= \frac{1}{2\pi}\int_{2\pi} |H(\omega)|^2 S_{xx}(\omega)d\omega$$

Example 4

- **Find response of system *h*(*n*) to *x*(*n*):**

$$h(n)=\left(\frac{1}{2}\right)^n u(n) \qquad x(n)=\left(\frac{1}{4}\right)^n u(n)$$

- **<u>Solution</u>:**

$$H(\omega)=\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}=\frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$X(\omega)=\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$Y(\omega)=H(\omega)X(\omega)$$

$$=\frac{1}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$S_{yy}(\omega)=\left|H(\omega)\right|^2 S_{xx}(\omega)$$

$$=\frac{1}{\left(\frac{5}{4}-\cos\omega\right)\left(\frac{17}{16}-\frac{1}{2}\cos\omega\right)}$$

Frequency Response of LTI Systems

$$H(\omega) = H(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

- **For rational transfer functions, we have:**

$$H(z) = \frac{A(z)}{B(z)}$$

coefficients

zeros

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k z^{-1}\right)}{\prod_{k=1}^{N}\left(1 - p_k z^{-1}\right)}$$

poles

$$H^*\left(1/z^*\right) = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k z^*\right)^*}{\prod_{k=1}^{N}\left(1 - p_k z^*\right)^*} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k^* z\right)}{\prod_{k=1}^{N}\left(1 - p_k^* z\right)}$$

$$H(\omega) = \frac{B(\omega)}{A(\omega)}$$

$$H(\omega) = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1 - p_k e^{-j\omega}\right)}$$

$$H^*(\omega) = H^*\left(1/z^*\right)\big|_{z=e^{j\omega}}$$

$$= b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k e^{-j\omega}\right)^*}{\prod_{k=1}^{N}\left(1 - p_k e^{-j\omega}\right)^*} = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k^* e^{j\omega}\right)}{\prod_{k=1}^{N}\left(1 - p_k^* e^{j\omega}\right)}$$

Frequency Response of LTI Systems

$$\left|H(\omega)\right|^2 = H(\omega)H^*(\omega)$$

- **For real coefficients a_k's and b_k's (i.e., when $h(n)$ is real):**
  - Zeros (z_k's) and poles (p_k's) are either real or occur in complex conjugate pairs
- **Therefore, we have**

$$H^*\left(1/z^*\right) = H\left(z^{-1}\right) \qquad \text{and} \qquad H^*(\omega) = H(-\omega)$$

- **This implies**

$$\left|H(\omega)\right|^2 = H(\omega)H^*(\omega) = H(\omega)H(-\omega) = H(z)H\left(z^{-1}\right)\Big|_{z=e^{j\omega}}$$

- **We know that**

$$r_{hh}(l) = h(l) * h(-l) \quad \Rightarrow \quad R_{hh}(z) = H(z)H\left(z^{-1}\right)$$

$$\Rightarrow \left|H(\omega)\right|^2 = \text{Fourier Transform}\left\{r_{hh}(l)\right\}$$

Example 5

- **Determine $|H(\omega)|^2$ for the system:**

$$y(n) = -0.1y(n-1) + 0.2y(n-2) + x(n) + x(n-1)$$

- **<u>Solution</u>:**

$$H(z) = \frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}}, \qquad \text{ROC}: |z| > 0.5$$

  – ROC includes unit circle, hence $H(\omega)$ exists

$$H(z)H\left(z^{-1}\right) = \frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}} \cdot \frac{1+z}{1+0.1z-0.2z^{2}}$$

$$= \frac{2+z+z^{-1}}{1.05+0.08\left(z+z^{-1}\right)-0.2\left(z^{2}+z^{-2}\right)}$$

$$|H(\omega)|^2 = \frac{2(1+\cos\omega)}{1.45+0.16\cos\omega-0.8\cos^2\omega}$$

Computation of the Frequency Response Function

$N \geq M$

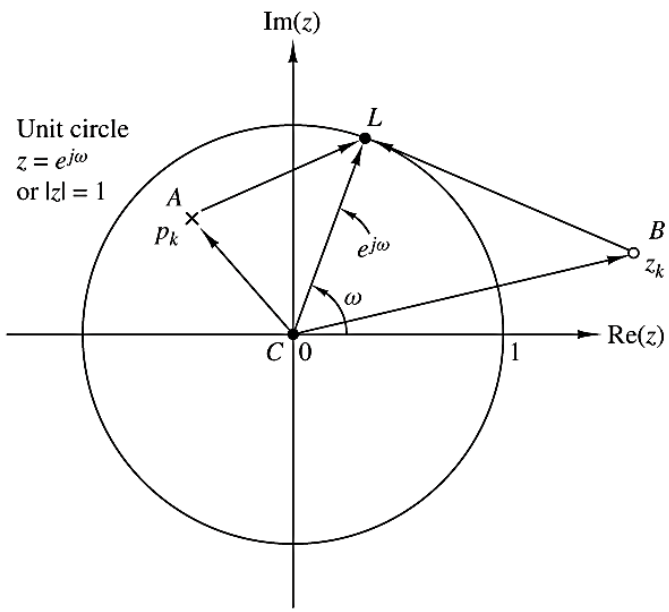
$$H(\omega) = b_0 \frac{\prod_{k=1}^{M}\left(1 - z_k e^{-j\omega}\right)}{\prod_{k=1}^{N}\left(1 - p_k e^{-j\omega}\right)} = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}\left(e^{j\omega} - z_k\right)}{\prod_{k=1}^{N}\left(e^{j\omega} - p_k\right)}$$

- **Express in polar form:**

$$e^{j\omega} - z_k = V_k(\omega) e^{j\Theta_k(\omega)}$$

$$e^{j\omega} - p_k = U_k(\omega) e^{j\Phi_k(\omega)}$$

Evaluate $H(w)$ at w
⇔ Evaluate $H(z)$ at point L



$$\mathbf{CL} = e^{j\omega}, \mathbf{CA} = p_k \qquad \mathbf{CL} = e^{j\omega}, \mathbf{CB} = z_k$$

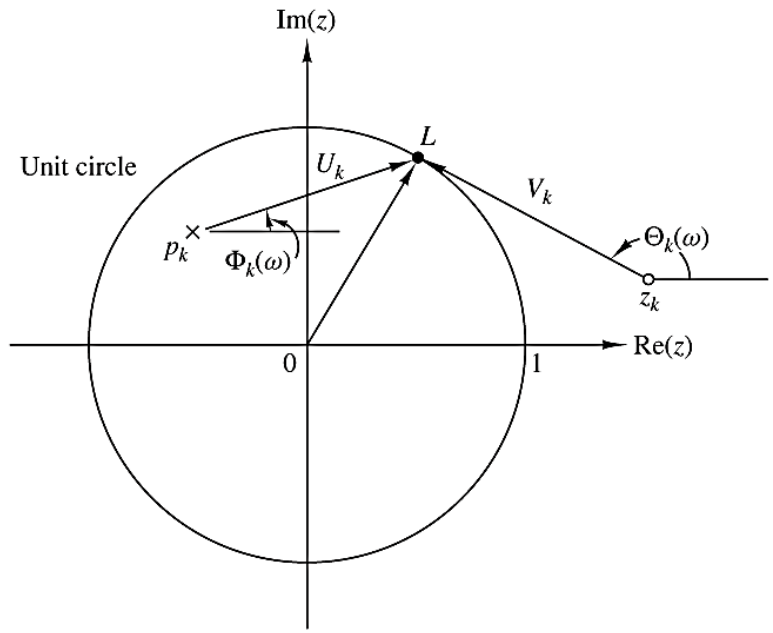
$$\mathbf{CL} = \mathbf{CA} + \mathbf{AL} \qquad \mathbf{CL} = \mathbf{CB} + \mathbf{BL}$$

$$\mathbf{AL} = e^{j\omega} - p_k = U_k(\omega) e^{j\Phi_k(\omega)}$$

$$\mathbf{BL} = e^{j\omega} - z_k = V_k(\omega) e^{j\Theta_k(\omega)}$$

Computation of the Frequency Response Function

- **Phase interpretation:**



Geometric Interpretation of Poles and Zeros

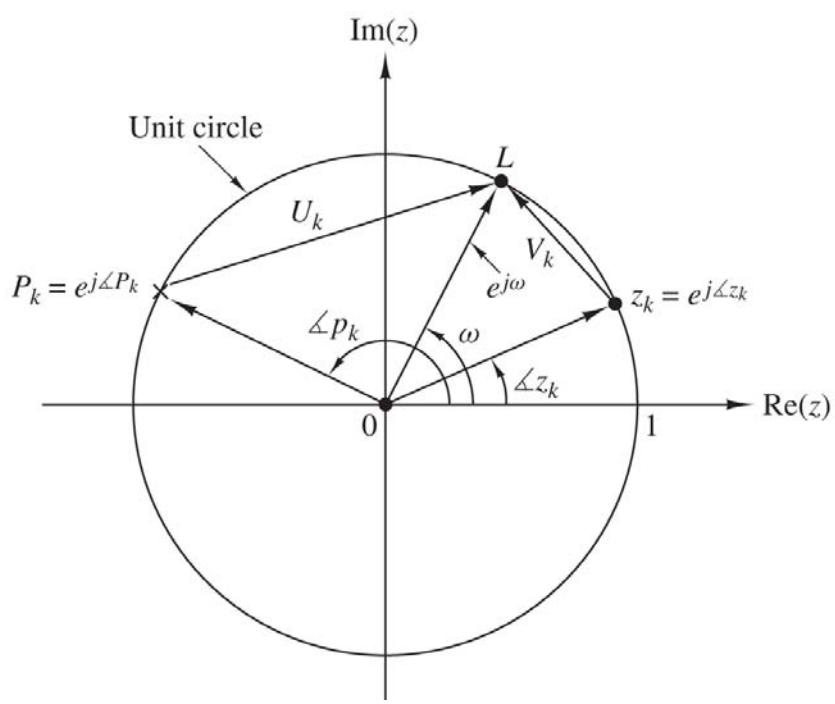
- **Assume a zero z_k and a pole p_k are on the unit circle**

  $At\ \omega = \measuredangle z_k, V_k(\omega)$ and consequently $|H(\omega)|$ become zero

  $At\ \omega = \measuredangle p_k, U_k(\omega)$ becomes 0 and consequently $|H(\omega)|$ becomes ∞
- **Presence of a zero close to unit circle causes $|H(\omega)|$ at frequencies that correspond to points on unit circle close to that point, to be small**
- **Poles have opposite effect**
- **Placing a pole close to a zero cancels effect of zero, and vice versa**

$$e^{j\omega} - z_k = V_k(\omega)e^{j\Theta_k(\omega)}$$

$$e^{j\omega} - p_k = U_k(\omega)e^{j\Phi_k(\omega)}$$



Example 6

- **Find the frequency response of:**

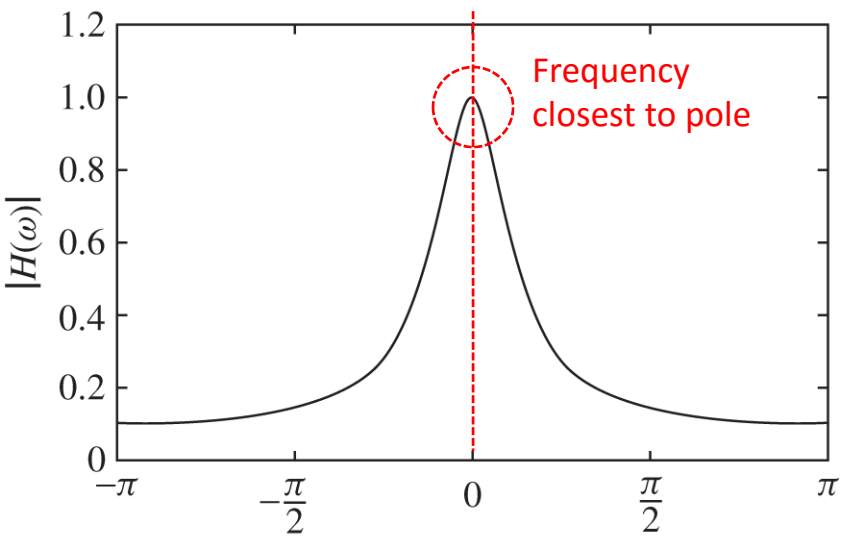
$$H(z) = \frac{1}{1-0.8z^{-1}} = \frac{z}{z-0.8}$$

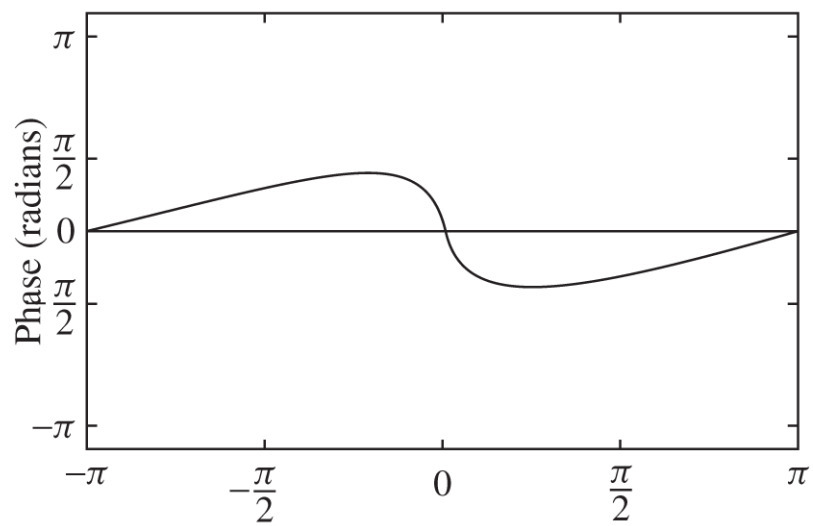
- **Solution: *H* has**
  - Zero at $z = 0$
  - Pole at $z = 0.8$

$$H(\omega) = \frac{e^{j\omega}}{e^{j\omega}-0.8}$$

$$|H(\omega)| = \frac{1}{\sqrt{1.64-1.6\cos\omega}}$$

$$\theta(\omega) = \omega - \tan^{-1}\frac{\sin\omega}{\cos\omega - 0.8}$$





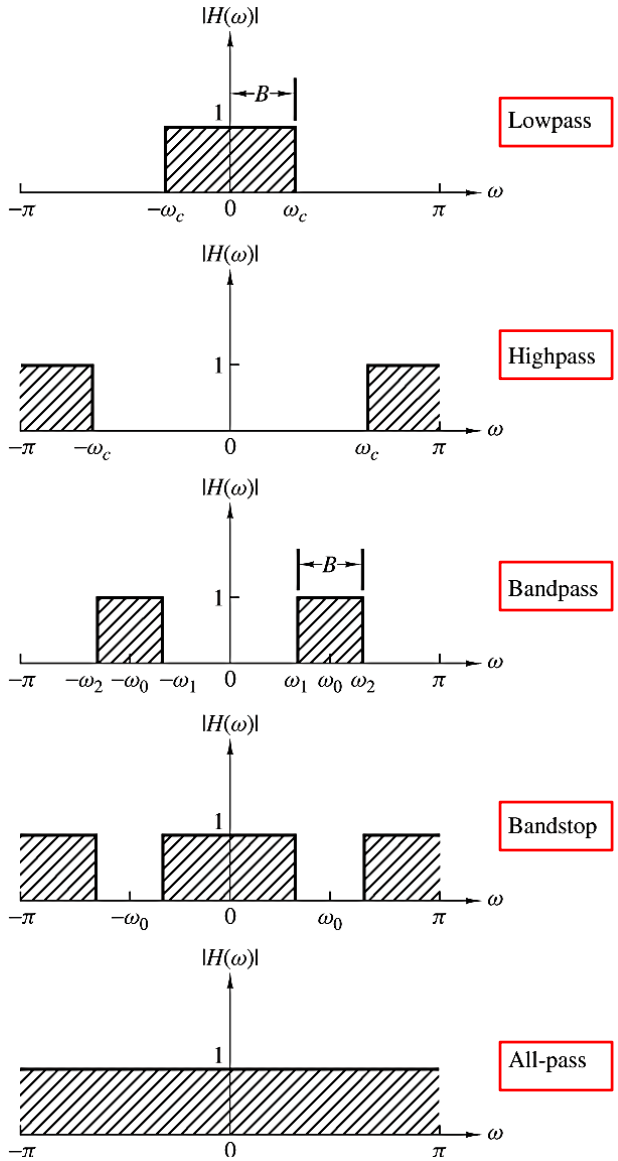
Part III: LTI Systems as Frequency Selective Filters

Ideal Frequency Selective Discrete-Time Filters

$$Y(\omega) = H(\omega)X(\omega)$$

Magnitude responses

What about phase?



Phase Response of Ideal Filters

- **An ideal filter has *linear* phase response. Why?**
- **Assume a signal is passed through a filter with frequency response:**

$$H(\omega) = \begin{cases} Ce^{-j\omega n_0} & -\omega_1 \le \omega \le \omega_2 \\ 0 & \text{elsewhere} \end{cases}$$

C, n_0 constants $\longrightarrow$ Phase: $-wn_0$

$$Y(\omega) = H(\omega)X(\omega)$$
$$= CX(\omega)e^{-j\omega n_0}, \quad -\omega_1 \le \omega \le \omega_2$$

$$y(n) = Cx(n - n_0)$$

Linear phase translates into delay

This is tolerable and not considered distortion

- **In general, linear phase** $\Theta(\omega) = -\omega n_0$
- **The derivative of** $\Theta(\omega)$ **is called group delay:**

$$\tau_g(\omega) = -\frac{d\Theta(\omega)}{d\omega}$$

  – Time delay that a signal component at frequency *w* undergoes as it passes through filter