EECE 491: Discrete-time Signal Processing

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Lecture 4: Z-Transform

Announcements

- **Reading**
  - Chapter 3, Proakis

The Z-Transform

- **Used for:**
  - Analysis of LTI systems
  - Solving difference equations
  - Determining system stability
  - Finding frequency response of stable systems

Eigen Functions of LTI System

- **Consider an LTI system with an impulse response *h*[*n*]:**

- **Let** $x[n] = e^{j\omega_0 n}$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 (n-k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k] e^{j\omega_0 k} \right) \times e^{j\omega_0 n}$$

$$= H\left(e^{j\omega}\right)\Big|_{\omega=\omega_0} \times e^{j\omega_0 n}$$

$$H\left(e^{j\omega}\right) = \mathrm{DTFT}\{h[n]\}$$

- **Output is the same pure frequency, scaled and phase-shifted!**
- **Therefore,** $e^{j\omega_0 n}$ **is an Eigen-function of an LTI system**

Eigen Functions of LTI System

- **What if** $x[n] = z^n = re^{j\omega_0 n}$ **?**

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} h[k] z^{n-k} \\ &= \left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right) z^n = H(z) z^n \end{aligned}$$

- $x[n] = z^n$ **is also an Eigen-function of an LTI system**
- **Transfer function:**

$$\boxed{H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}}$$

- ***H*(*z*) exists for larger class of *h*[*n*] than** $H\left(e^{j\omega}\right)$

The Z Transform

- **Bilateral Z-Transform:**

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

- **Since** $z = re^{j\omega}$

$$\boxed{X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \mathrm{DTFT}\{x[n]\}}$$

- **Region of Convergence (ROC)**
  - Set of values of *z* for which the sum converges

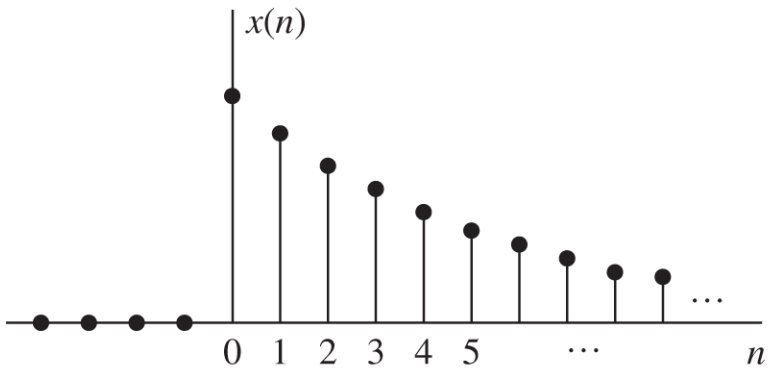
$$\boxed{\sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

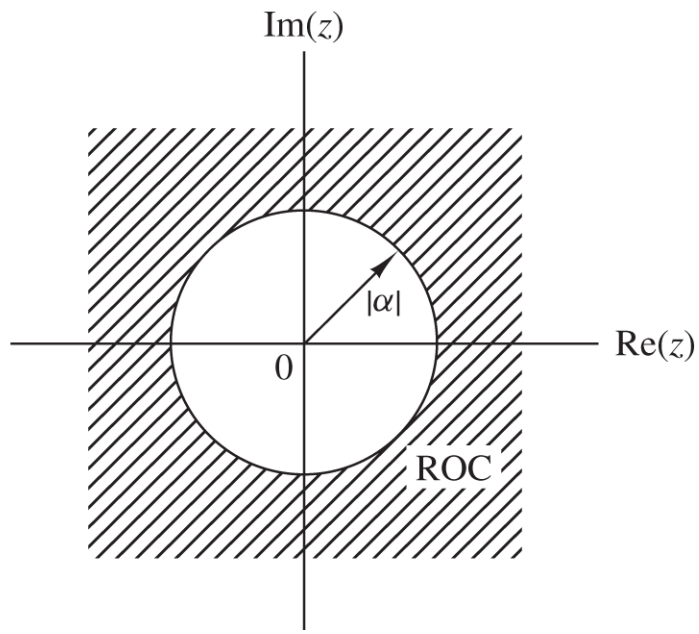
Example 1

- **Right-sided sequence** $x[n] = \alpha^n u[n]$ **:**

$$X(z) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \left(\alpha z^{-1}\right)^n$$

$$= \frac{1}{1-\alpha z^{-1}} \qquad \text{ROC} = \left\{ z \ : \ |z| > |\alpha| \right\}$$





Example 2

- **Left-handed sequence:** $x[n] = -\alpha^n u[-n-1]$

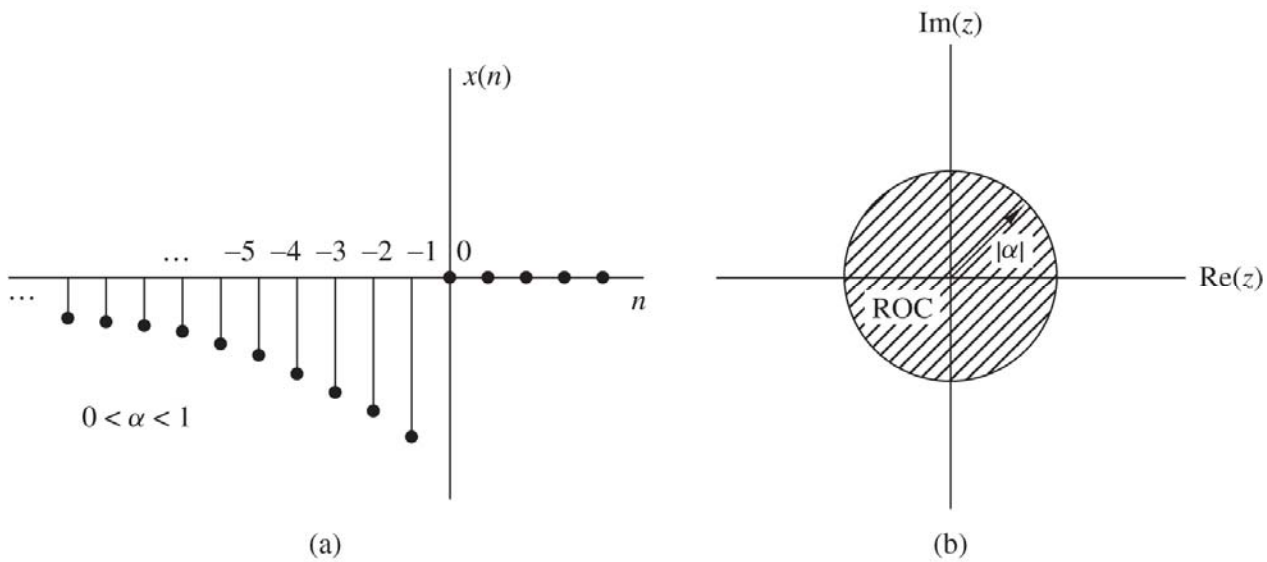
$$X(z) = -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = \sum_{n=1}^{\infty} \left(\alpha^{-1} z\right)^n$$

$$= \frac{\alpha^{-1}}{1-\alpha^{-1} z^{-1}}$$

$$= \frac{1}{1-\alpha z^{-1}}$$

$$\text{ROC} = \left\{ z \ : \ |z| < |\alpha| \right\}$$

The Z-transform without the ROC does not uniquely define a sequence!



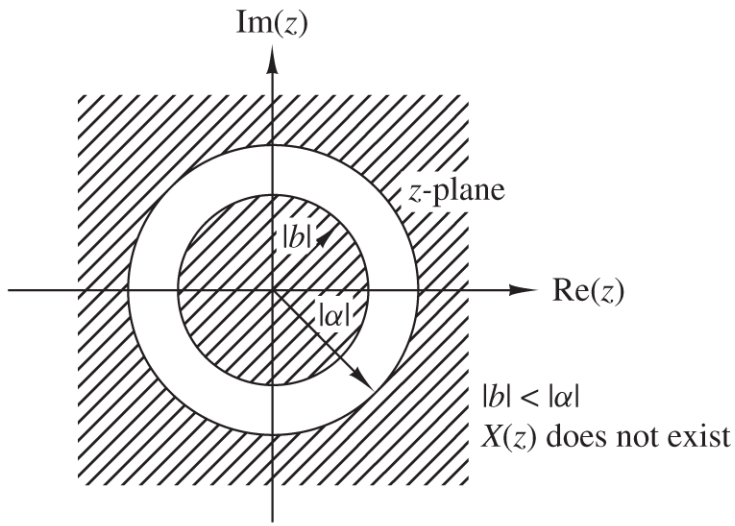
Example 3

- **Two-sided sequence:** $x[n] = \alpha^n u[n] + b^n u[-n-1]$

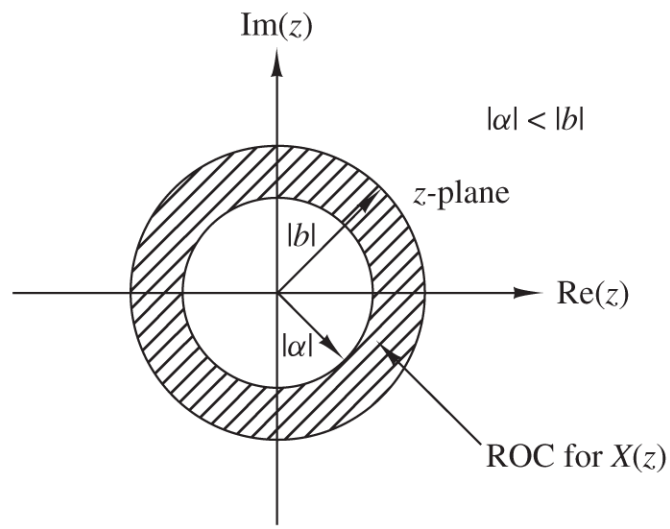
$$X(z) = \sum_{n=0}^{\infty} \left(\alpha z^{-1}\right)^n + \sum_{n=1}^{\infty} \left(b^{-1} z\right)^n$$

- **If $|b|<|\alpha|$: the two ROC's do not overlap and z-Transform does not exist**
- **If $|b|>|\alpha|$: ROC is a ring and $X(z)$ exists**

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-bz^{-1}} \qquad \text{ROC} = \left\{ z \ : \ |\alpha| < |z| < |b| \right\}$$



(a)



(b)

Example 4

- **Let**

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{3}\right)^n u[-n-1]$$

- **Then, ROC is**

$$\begin{aligned} \text{ROC} &= \left\{ z \ : \ |z| > \frac{1}{2} \right\} \ \cap \ \left\{ z \ : \ |z| < \frac{1}{3} \right\} \\ &= \phi \end{aligned}$$

Example 5

- **Let** $x[n] = a^n, \quad a \neq 0$

$$\begin{aligned} \text{ROC} &= \{z \ : \ |z| > a\} \ \cap \ \{z \ : \ |z| < a\} \\ &= \phi \end{aligned}$$

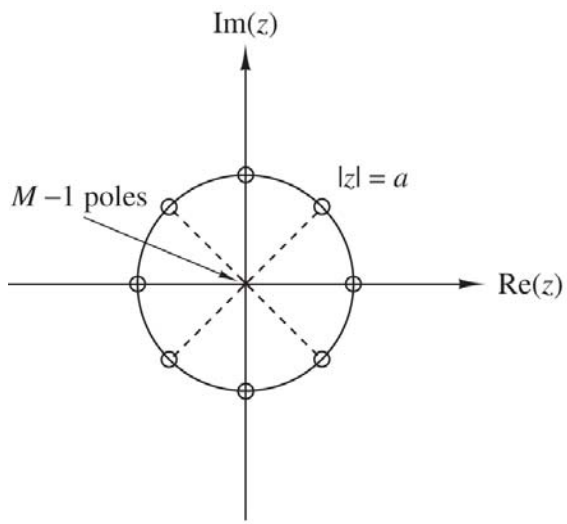
Example 6

- **Finite sequence:** $x[n] = a^n u[n] u[-n+M-1]$

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n} \qquad \text{Finite, always converges}$$

$$= \frac{1-a^M z^{-M}}{1-az^{-1}} \qquad \text{zero cancels the pole}$$

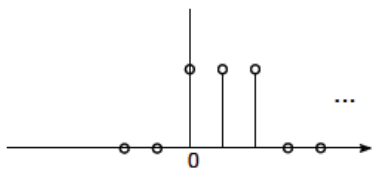
$$= \prod_{k=1}^{M-1} \left(1 - ae^{j\frac{2\pi k}{M}} z^{-1}\right) \qquad \text{ROC} = \{z \ : \ |z| > 0\}$$



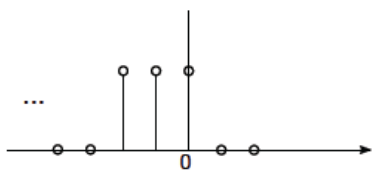
Pole–zero pattern for the finite-duration signal $x(n) = a^n$, $0 \le n \le M-1 (a > 0)$, for $M = 8$.

Properties of the ROC

- **P1: A ring or a disk in the Z-plane, centered at the origin**
- **P2: DTFT converges iff ROC includes the unit circle**
- **P3: ROC cannot contain poles**
- **P4: For finite duration sequences, ROC is the entire Z-plane, except possibly *z* = 0 or *z* = 1**



$X(z) = 1 + z^{-1} + z^{-2}$ ROC excludes $z = 0$



$X(z) = 1 + z^{1} + z^{2}$ ROC excludes $z = \infty$

Properties of the ROC

- **P5: For right-sided sequences ($x[n] = 0$ for all $n < N_1$, for some N_1) ROC extends outward from the outer-most pole to infinity.**

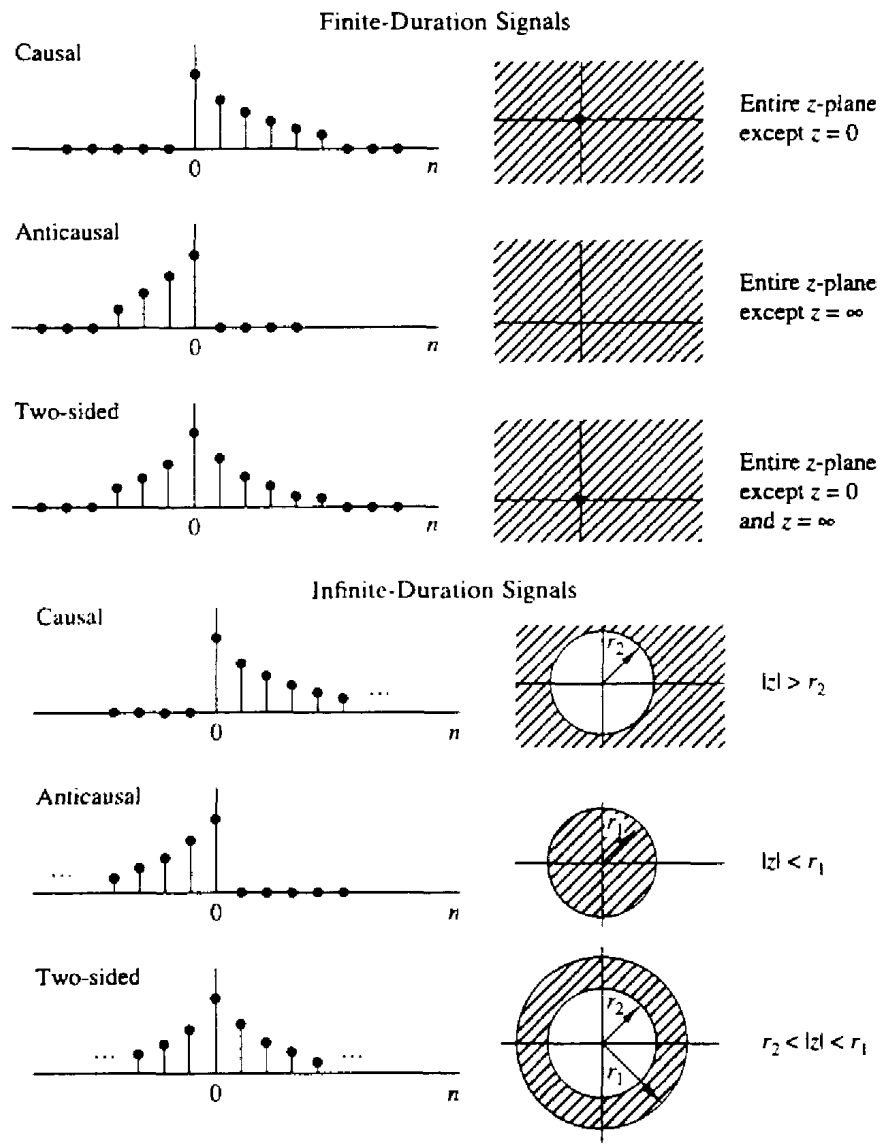
- **P6: For left-sided sequences ($x[n] = 0$ for all $n > N_2$, for some N_2) ROC extends inwards from the inner-most pole to zero.**

- **P7: For two-sided sequences, ROC is a ring**
  - Inner-bound: Pole with **largest** magnitude that contributes for $n > 0$
  - Outer-bound: Pole with **smallest** magnitude that contributes for $n < 0$

Characteristic Families of Signals with their ROCs

finite

infinite



Several Properties of the Z Transform

$$x[n-n_d] \quad \leftrightarrow \quad z^{-n_d}X(z)$$

$$z_0^n x[n] \quad \leftrightarrow \quad X(\tfrac{z}{z_0})$$

$$nx[n] \quad \leftrightarrow \quad -z\frac{dX(z)}{dz}$$

$$x[-n] \quad \leftrightarrow \quad X(z^{-1})$$

$$x[n] * y[n] \quad \leftrightarrow \quad X(z)Y(z) \qquad \text{ROC at least } \mathrm{ROC}_x \cap \mathrm{ROC}_y$$

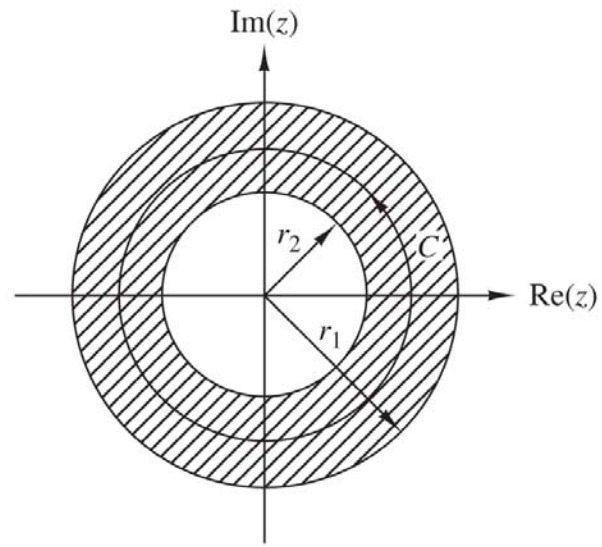
Some Common z-Transform Pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Inversion of the Z-Transform

- **In general, we can invert by contour integration within the ROC:**

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1}$$



- **Ways to avoid it:**
  - Inspection (known transforms)
  - Properties of the Z-transform
  - Power series expansion
  - Partial fraction expansion
  - Residue theorem

- **Most useful is the inverse of rational polynomials**

$$X(z) = \frac{B(z)}{A(z)}$$

Causal LTI Systems

- **Described by linear constant-coefficient difference equations:**

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=0}^{M} a_k z^{-k}}$$

Inversion of the Z-Transform

Example: Long division

$$X(z) = \frac{2 + z^{-1}}{1 - \frac{1}{2}z^{-1}}, \qquad \text{ROC} = \{z : \ |z| > \frac{1}{2}\}$$

x[n] right/left sequencs?

Arrange num/denum in ascending powers of z^{-1}

What about ROC $= \{z : \ |z| < \frac{1}{2}\}$?

$$
\begin{array}{r|l}
 & 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \cdots \\
\hline
1 - \frac{1}{2}z^{-1} & 2 + z^{-1} \\
 & 2 - z^{-1} \\
\cline{2-2}
 & \quad 2z^{-1} \\
 & \quad 2z^{-1} - z^{-2} \\
\cline{2-2}
 & \qquad z^{-2} \\
 & \qquad z^{-2} - \frac{1}{2}z^{-3} \\
\cline{2-2}
 & \qquad\quad \frac{1}{2}z^{-3}
\end{array}
$$

Inversion of the Z-Transform

$$
\begin{aligned}
X(z) &= 2 + 2z^{-1} + z^{-2} + \frac{1}{2}z^{-3} + \ldots \\
&= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\
x[n] &= 2\delta[n] + 2\delta[n-1] + \delta[n-2] + \frac{1}{2}\delta[n-3] + \ldots
\end{aligned}
$$

Inversion of the Z-Transform

- **Example: Partial Fraction Expansion**

$$
\begin{aligned}
X(z) &= \frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} \\
&= \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \\
&= \frac{A_1}{1-\frac{1}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{2}z^{-1}}
\end{aligned}
$$

Find A_1 and A_2

$$
\begin{aligned}
A_1 &= (1-\frac{1}{4}z^{-1})X(z)\Big|_{z=\frac{1}{4}} = -1 \\
A_2 &= (1-\frac{1}{2}z^{-1})X(z)\Big|_{z=\frac{1}{2}} = 2
\end{aligned}
$$

Inversion of the Z-Transform

Partial fraction expansion:

$$X(z) = \frac{-1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{2}z^{-1}}$$

From the tables:

$$x[n] = \left[-(\frac{1}{4})^n + 2(\frac{1}{2})^n\right] u[n] \qquad \text{because right sided}$$

Inversion of the Z-Transform

Partial fraction expansion in general:

$$X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

Suppose real and unrepeated poles $d_1, \cdots, d_N$

If $M < N$,

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$ Like example

Inversion of the Z-Transform

If $M \geq N$,

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

$$\Rightarrow \quad x[n] = \sum_{r=0}^{M-N} B_r \delta[n-r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

If d_k is a repeated pole of order S, replace $\frac{A_k}{1-d_k z^{-1}}$ with

$$\sum_{m=1}^{S} \frac{C_m}{(1 - d_k z^{-1})^m}$$

Example 7

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \qquad M = N = 2$$

$$= B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

Matching coefficients: $A_1 = -9,\ A_2 = 8,\ B_0 = 2$

$$\text{ROC} = \{z : \ |z| > 1\}$$

$$\Rightarrow \quad x[n] = 2\delta[n] - 9(\frac{1}{2})^n u[n] + 8u[n]$$

Time-Domain Behavior of a Single Real Pole Causal Signal

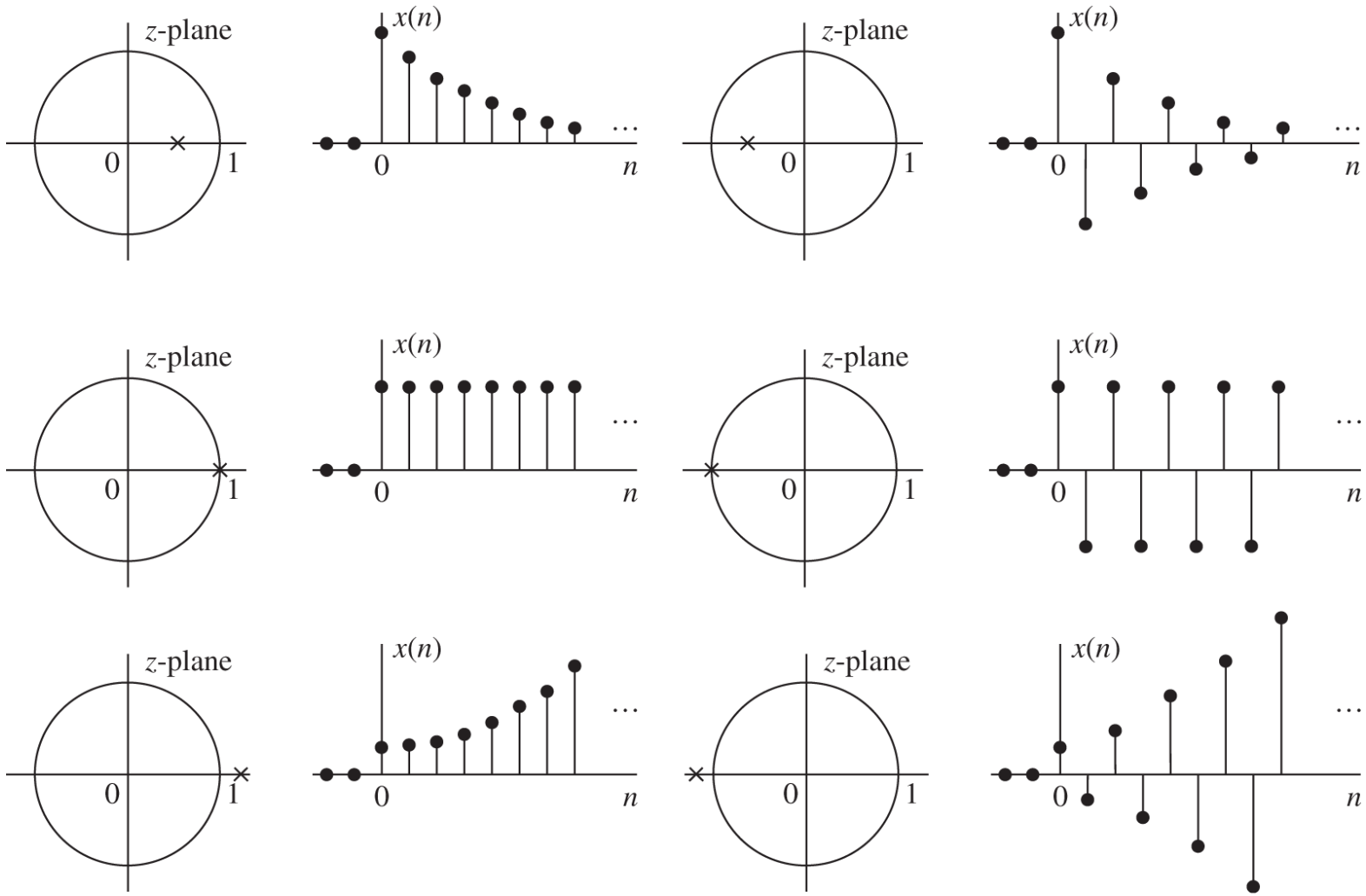


Figure 3.3.5 Time-domain behavior of a single-real-pole causal signal as a function of the location of the pole with respect to the unit circle.

Time-Domain Behavior of a Double Real Pole Causal Signal

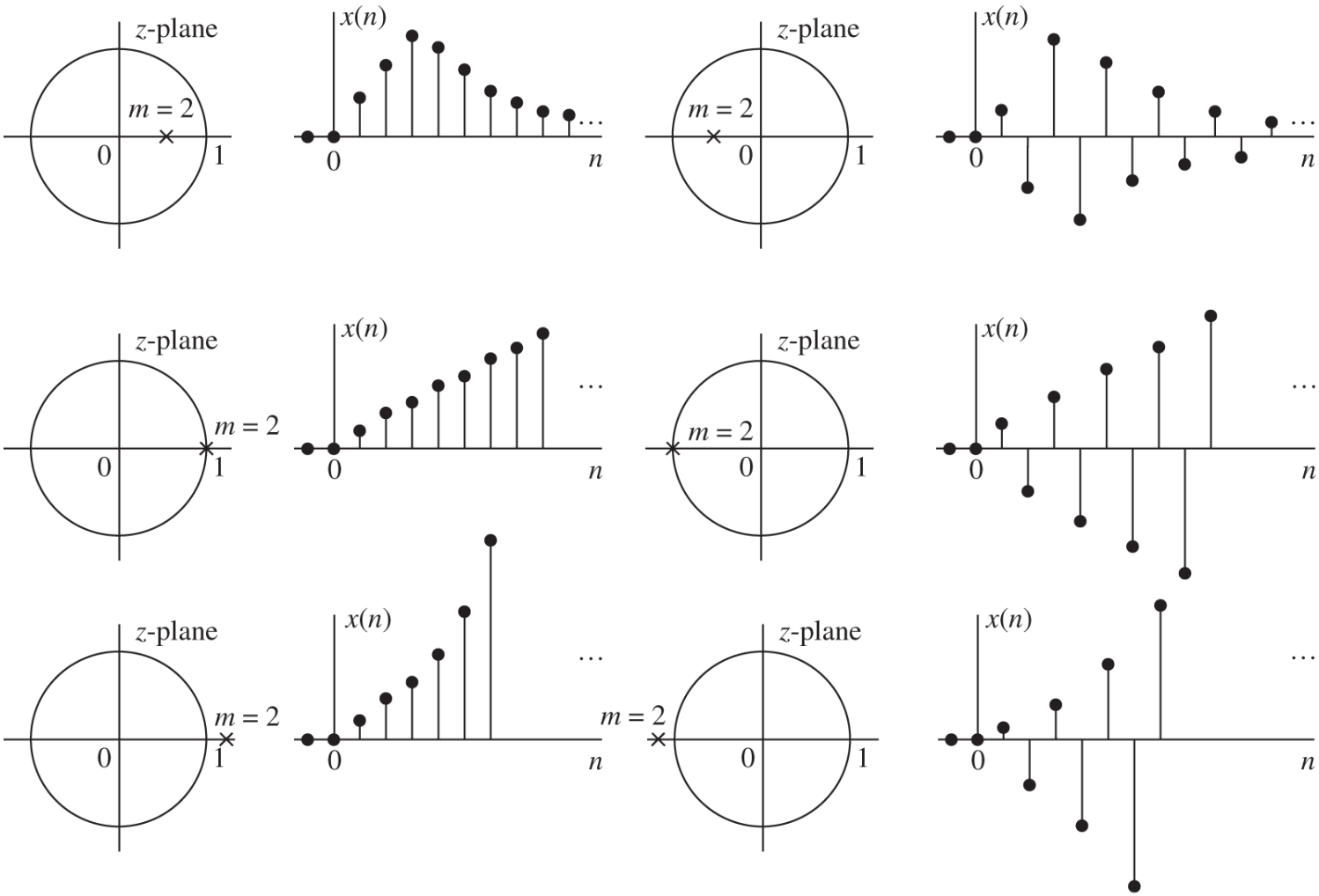


Figure 3.3.6 Time-domain behavior of causal signals corresponding to a double $(m = 2)$ real pole, as a function of the pole location.

Time-Domain Behavior of a Pair of Complex-Conjugate Poles

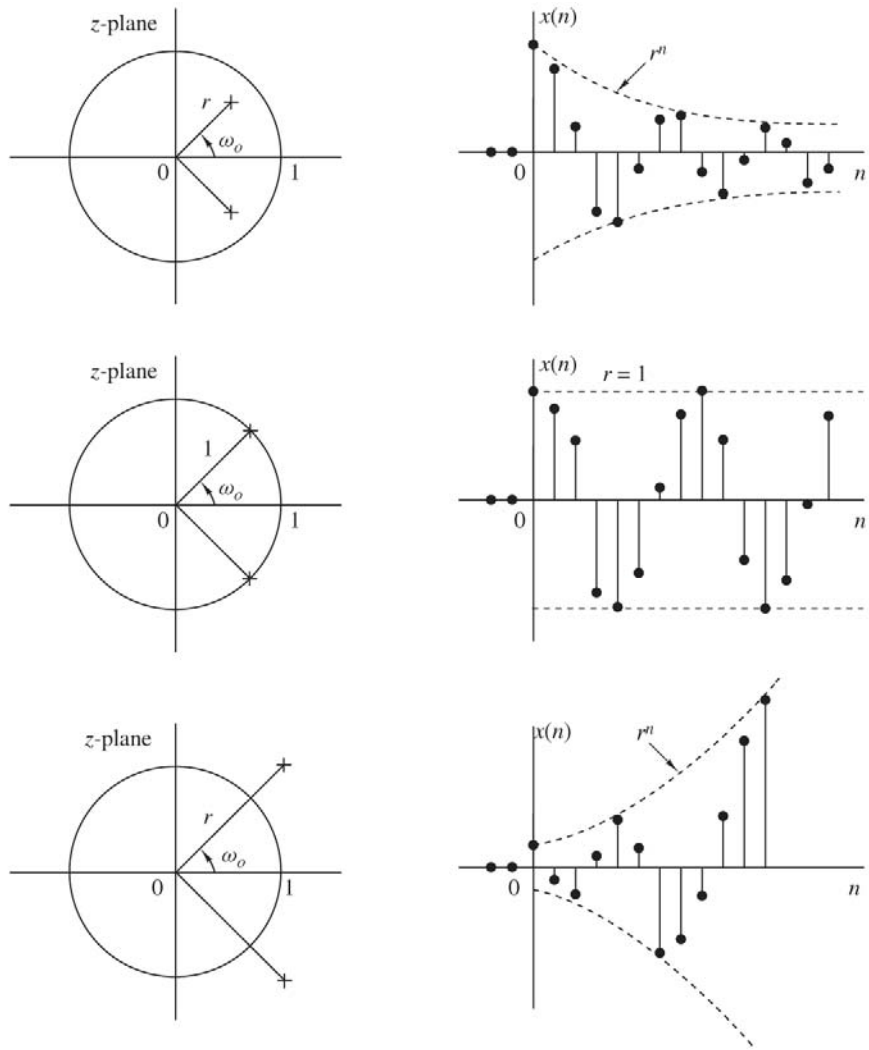
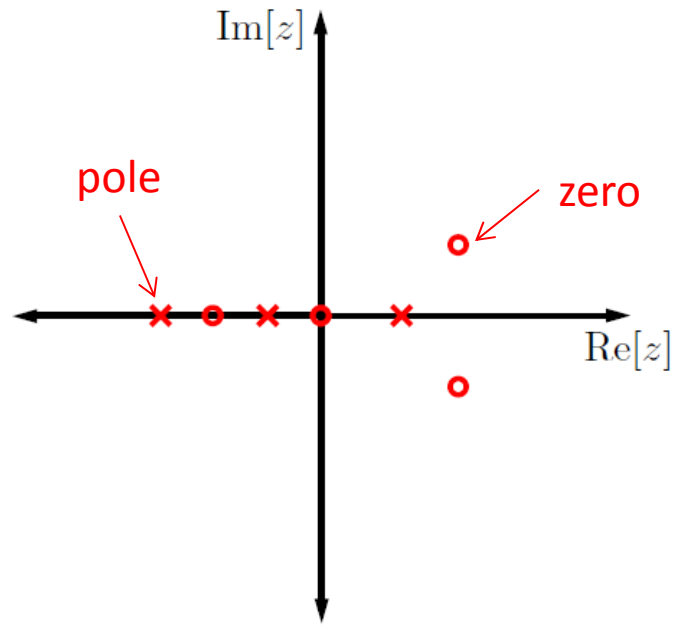


Figure 3.3.7 A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.

Stability and Causality of the z-Transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{m} b_k x[n-k]$$

$$H(z) = \frac{Y(z)}{X(z)}$$



- **Causal: ROC extends from outermost pole**
- **BIBO stable:**

$$|H(z)| = \left| \sum_{n=-\infty}^{\infty} h[n] z^{-n} \right|$$
$$\leq \sum_{n=-\infty}^{\infty} |h[n]| \left| z^{-n} \right|$$

- **Consider the unit circle $z = e^{jw}$. In this case we have $\left|z^{-n}\right| = \left|e^{-j\omega n}\right| = 1$. Then**

$$\left|H(e^{j\omega})\right| \leq \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- **Hence, BIBO stability → ROC of $H(z)$ includes the unit circle**