EECE 491: Discrete-time Signal Processing

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Lecture 3: Discrete-time Signal and Systems

Announcements

- **Reading**
  - Chapter 2, Proakis

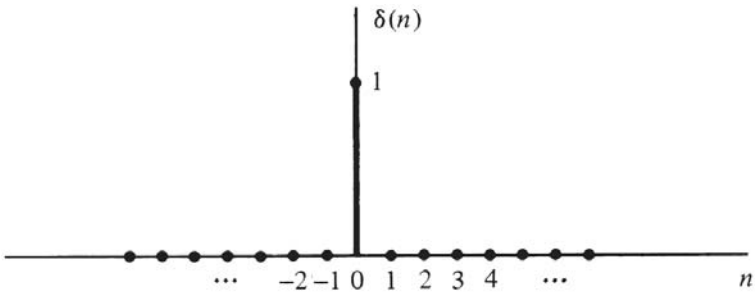
Outline

- DT Signals
- DT Systems
- Analysis of DT LTI Systems
- DT Systems Described by Difference Equations
- Implementation of DT Systems
- Correlation of DT Signals

I - Discrete-Time Signals

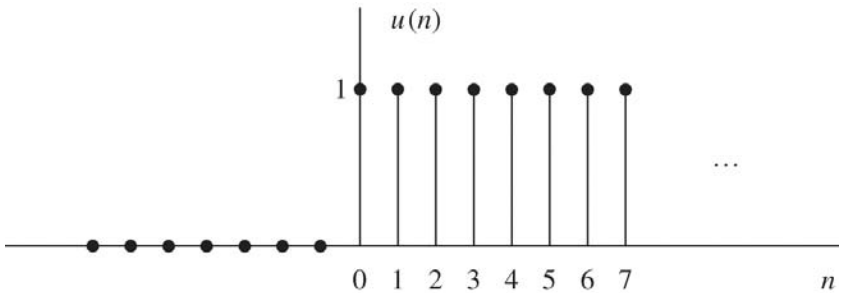
Elementary DT Signals: Unit Sample, Unit Step, Ramp Signals

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



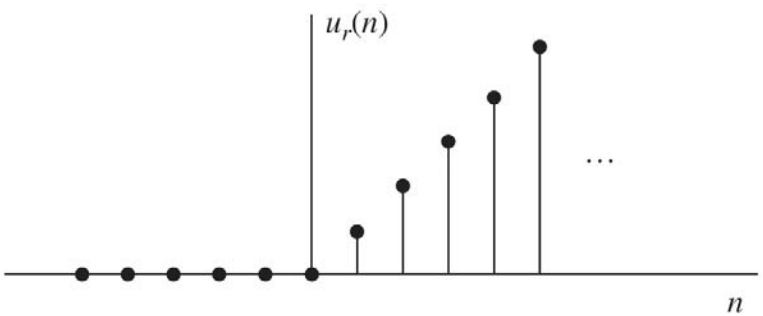
Unit Sample signal (Impulse)

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Unit Step signal

$$u_r(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

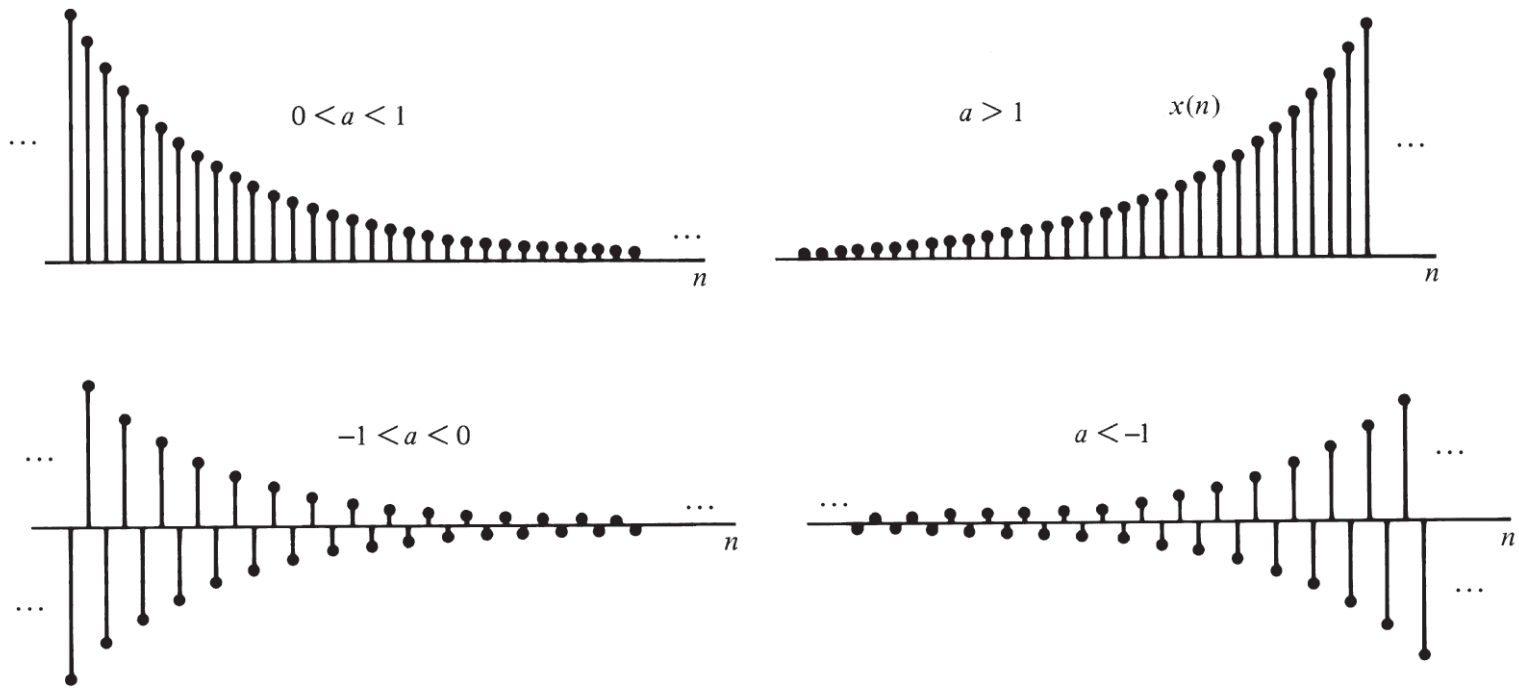


Ramp signal

Elementary DT Signals: Exponential

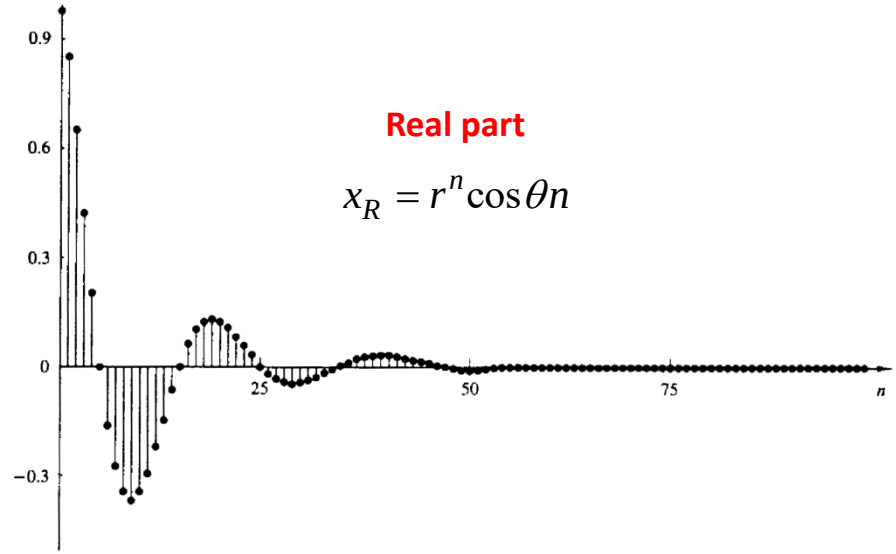
$$x(n) = a^n \quad \text{for all } n$$

Case a is real



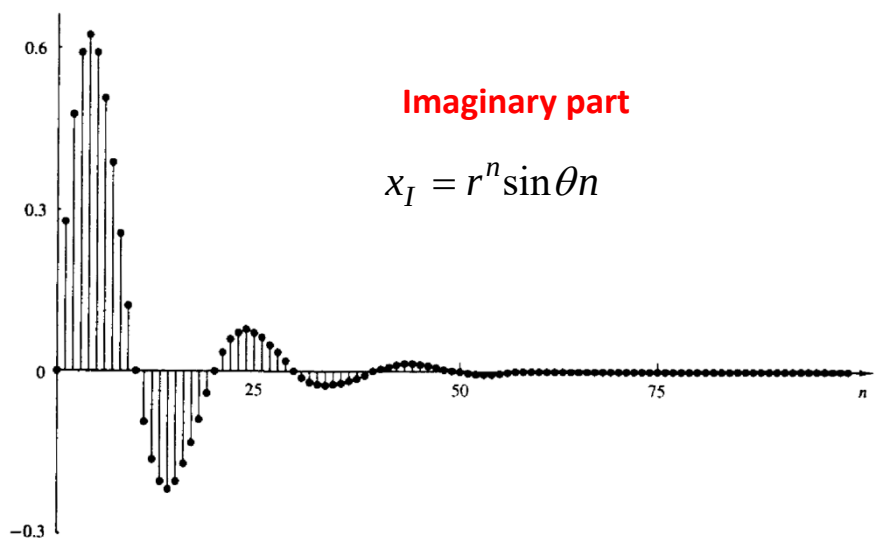
Elementary DT Signals: Exponential

$\boxed{\text{Case } a = re^{j\theta n} \text{ is complex}}$



Real part

$$x_R = r^n \cos\theta n$$



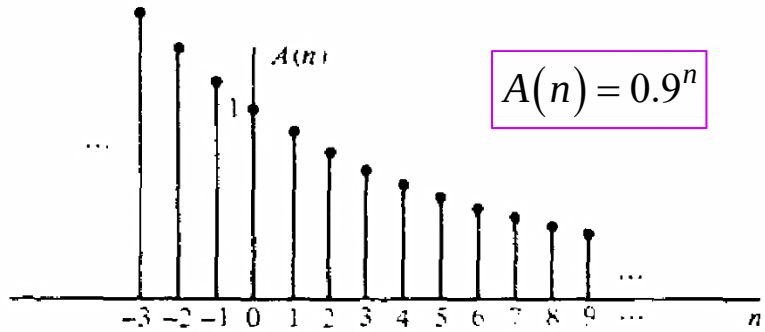
Imaginary part

$$x_I = r^n \sin\theta n$$

Elementary DT Signals: Exponential

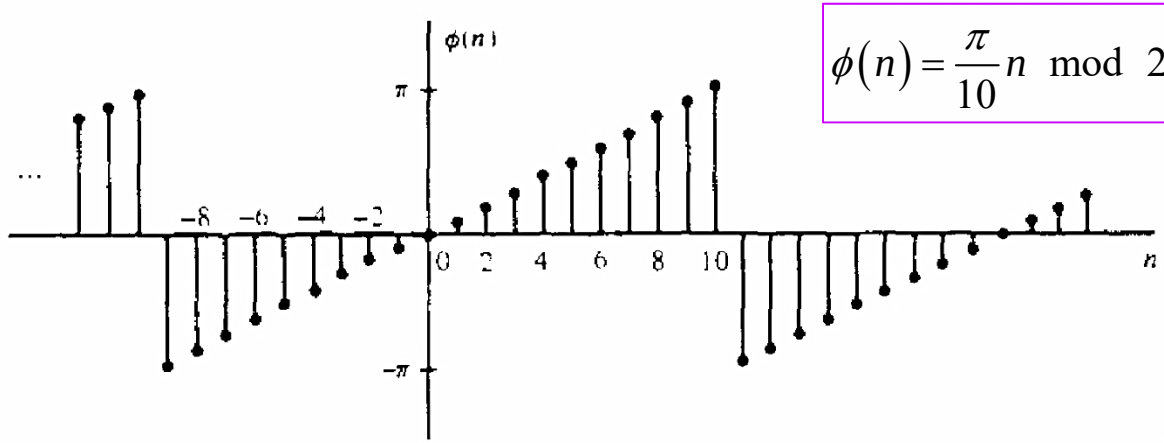
Case $x(n)$ is complex

$$x(n) = A(n)e^{j\phi(n)}$$



$$A(n) = 0.9^n$$

(a) Graph of $A(n) = r^n$, $r = 0.9$



$$\phi(n) = \frac{\pi}{10} n \mod 2\pi$$

Classification of DT Signals: Energy/Power

1. Energy signals and power signals

Energy:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E_N = \sum_{n=-N}^{N} |x(n)|^2$$

$$E = \lim_{N\to\infty} E_N$$

Average Power for $-N \le n \le N$:

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} E_N$$

- If *E* is finite, signal called Energy signal
- If *P* is finite, signal called Power signal

- **Many signals that posses infinite energy have finite average power**
  - Ex: Average power of unit step function $u(n)$ is

$$P == \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1 = \lim_{N\to\infty} \frac{N+1}{2N+1} = \frac{1}{2}$$

- **If *E* is finite, then $P = 0$**

Classification of DT Signals: Periodic/Aperiodic

2. **Periodic signals and aperiodic signals: x(n+N) = x(n) for all n**
   - Ex: $x(n) = A \sin(2\pi f_0 n)$, with $f_0 = k/N$, where N is the fundamental period

- **Energy of a periodic signal over one period $0 \le n \le N - 1$ is finite if $x(n)$ is finite**
- **Energy of a periodic signal for $-\infty < n < \infty$ is infinite**

- **Avg. power of a periodic signal is power over a single period N ($x(n)$ is finite):**

$$P = \frac{1}{N}\sum_{n=0}^{N-1}\left|x(n)\right|^2$$

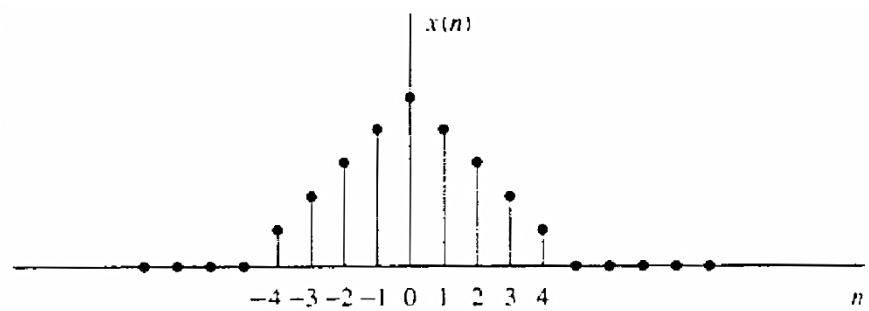
- **Hence, periodic signals are power signals**

Classification of DT Signals

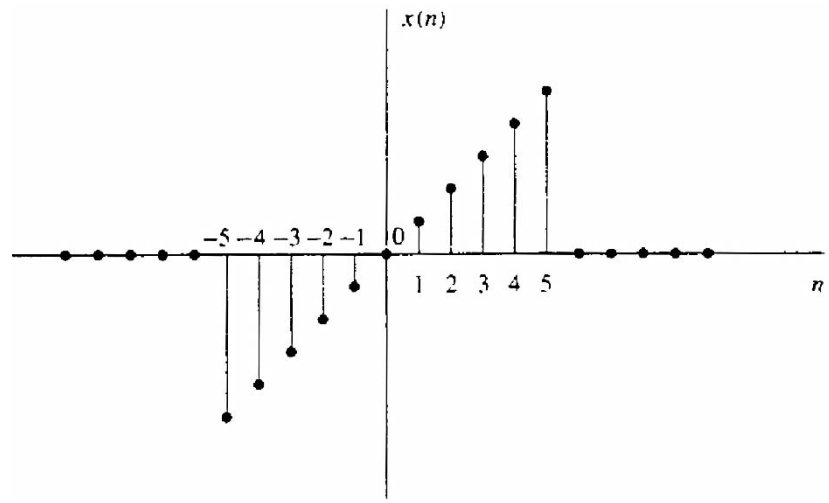
3. Symmetric (even) and anti-symmetric (odd) signals:

- Even iff: $x(-n) = x(n)$
- Odd iff: $x(-n) = -x(n)$. Hence $x(0) = 0$

Even



Odd



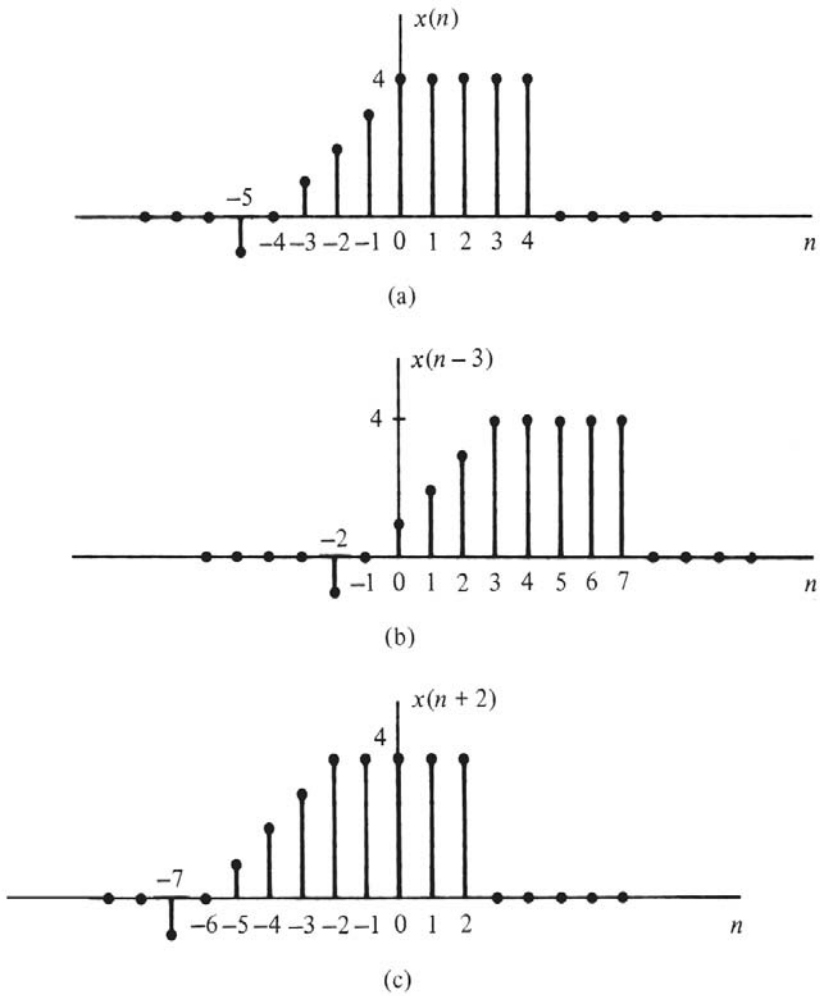
- **For an arbitrary signal x(n), we can express its even and odd parts as:**

$$x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

$$x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

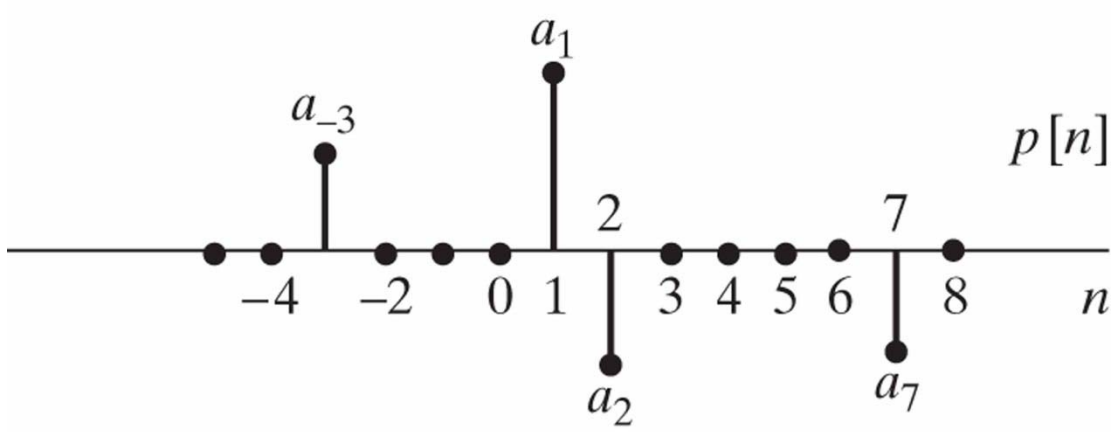
Time-Shifting Operation

- **Shifting by *k* time units:** $x(n) \rightarrow x(n-k)$



Time-Shifting Operation: Example

- **Consider the DT signal below. Write down an equation describing *p*(*n*)**

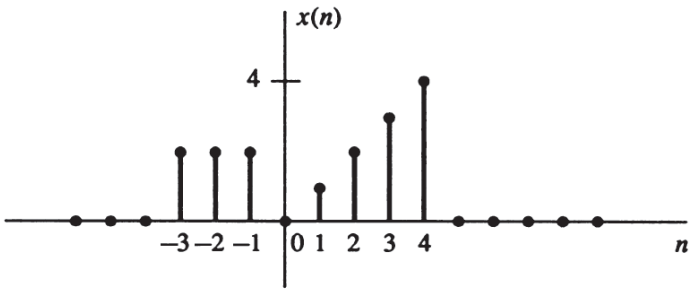


- ***Answer*: *p*(*n*) can be represented as a sum of 4 scaled and delayed impulses:**

$$p(n) = a_{-3} \cdot \delta(n+3) + a_1 \cdot \delta(n-1) + a_2 \cdot \delta(n-2) + a_7 \cdot \delta(n-7)$$

Folding Operation

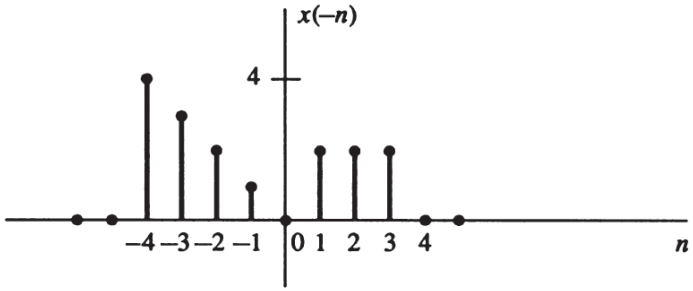
$$x(n) \rightarrow x(-n)$$



(a)

- **Note: Folding shifting are not commutative**
- **Fold then shift *x*(*n*) by *k*:**

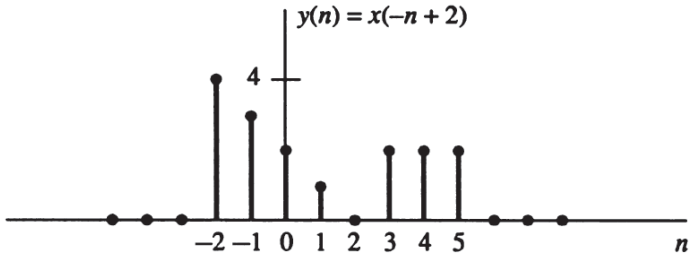
$$y_1(n) = x(-n+k)$$



(b)

- **Shift *x*(*n*) by *k* then fold:**

$$y_2(n) = x(-(n+k))$$

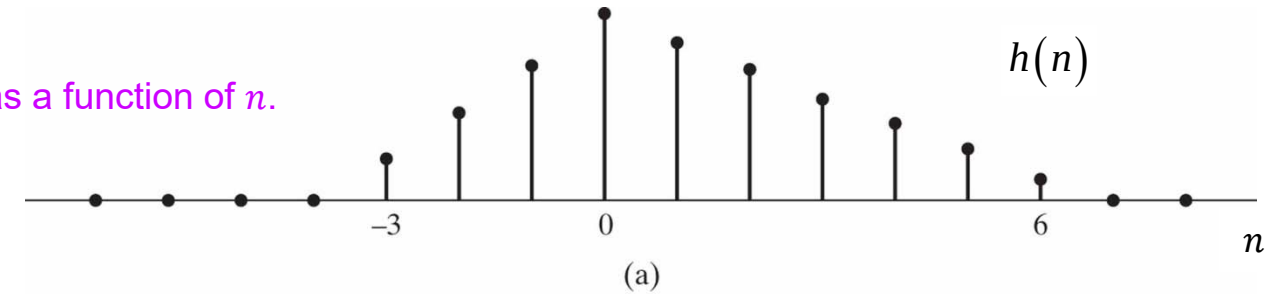


(c)

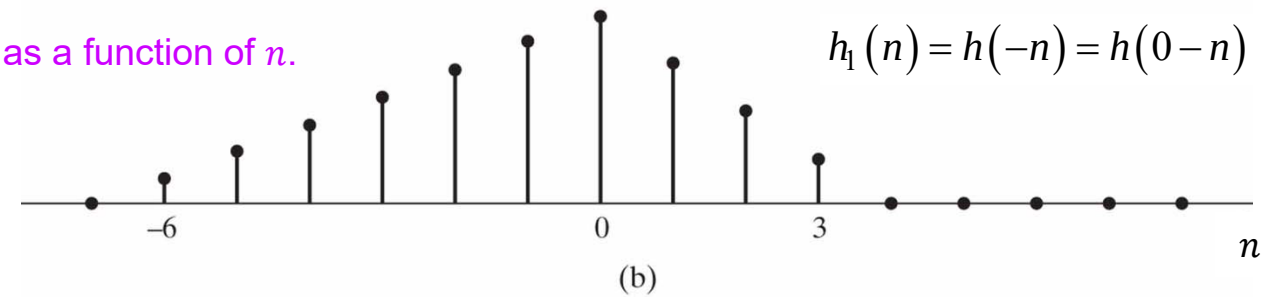
Time-Shifting and Folding: Example

- ***Example*: Form the sequence *h*(*k* – *n*) from *h*(*n*) .**

The sequence *h*(*n*) as a function of n.

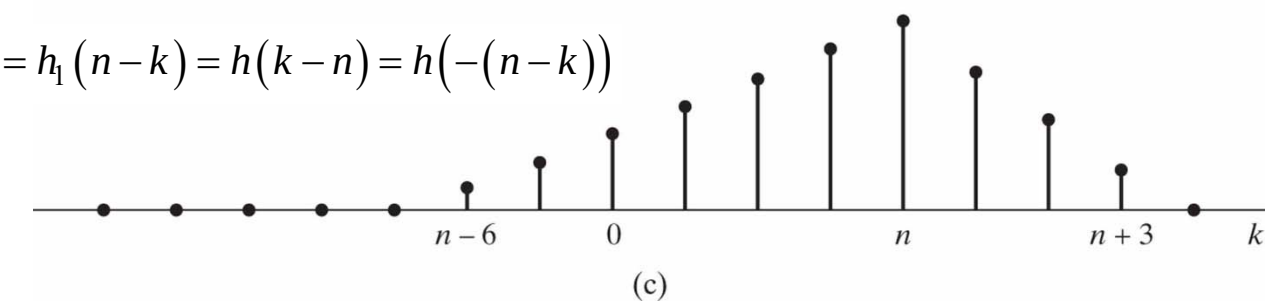


The sequence *h*(–*n*) as a function of n.



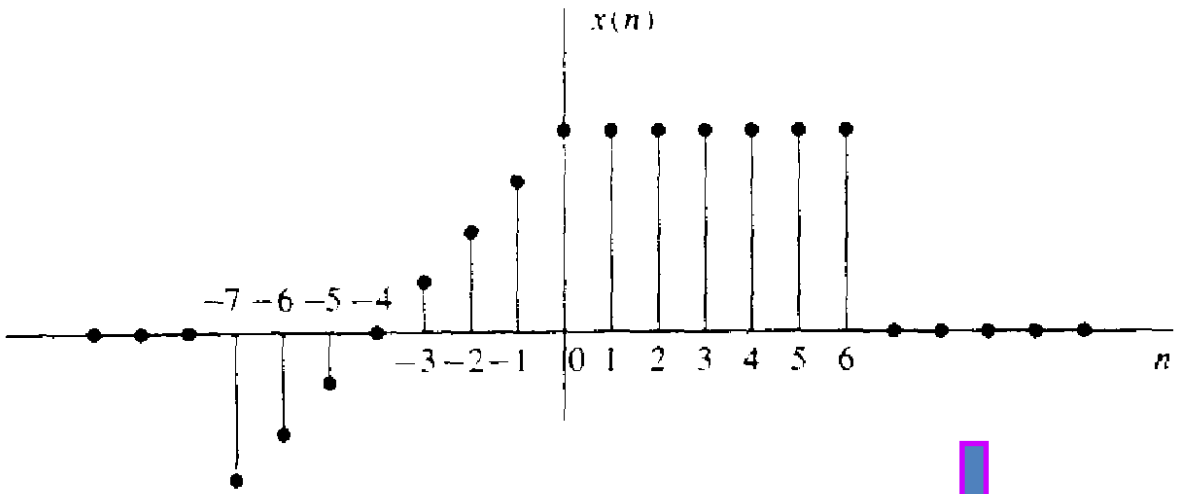
The sequence *h*(*k*–*n*) = *h*(–(*n* – *k*)) as a function of *n* for *k* = 4.

$$h_2(n) = h_1(n-k) = h(k-n) = h(-(n-k))$$



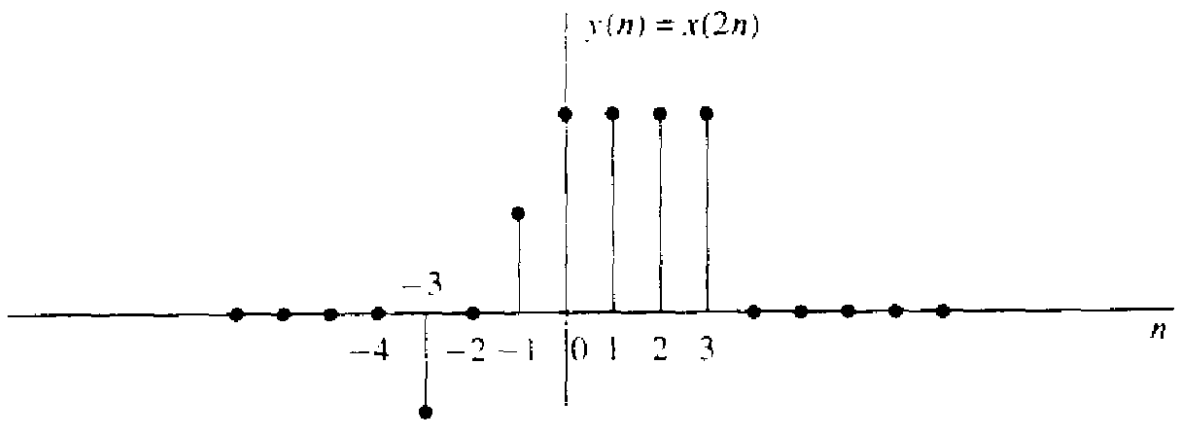
Down-Sampling Operation

- $x(n) \rightarrow x(\mu n)$ **where μ is an integer**



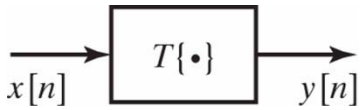
(a)

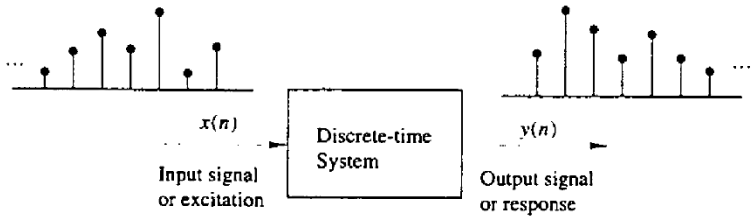
decrease sampling rate by 2x



Discrete-Time Systems

Input-Output Description of Systems





- ***Example*: Determine the response of the following systems to the input signal**

$$x(n)=\begin{cases}|n| & -3\le n\le 3\\ 0 & \text{otherwise}\end{cases}=\{\cdots 0,0,3,2,1,\hat{0},1,2,3,0,0,\cdots\}$$

$$y(n)=x(n+1)$$
$$A: y(n)=\{\cdots,3,2,1,0,\hat{1},2,3,0,\cdots\}$$

$$y(n)=\frac{1}{3}\left[x(n+1)+x(n)+x(n-1)\right]$$
$$A: y(n)=\left\{\cdots,0,1,\frac{5}{3},2,1,\frac{\hat{2}}{3},1,2,\frac{5}{3},1,0\cdots\right\}$$

(*moving average filter*)

$$y(n)=median\{x(n+1)+x(n)+x(n-1)\}$$
$$A: y(n)=\{0,2,2,1,1,\hat{1},2,2,0,0,0,\cdots\}$$

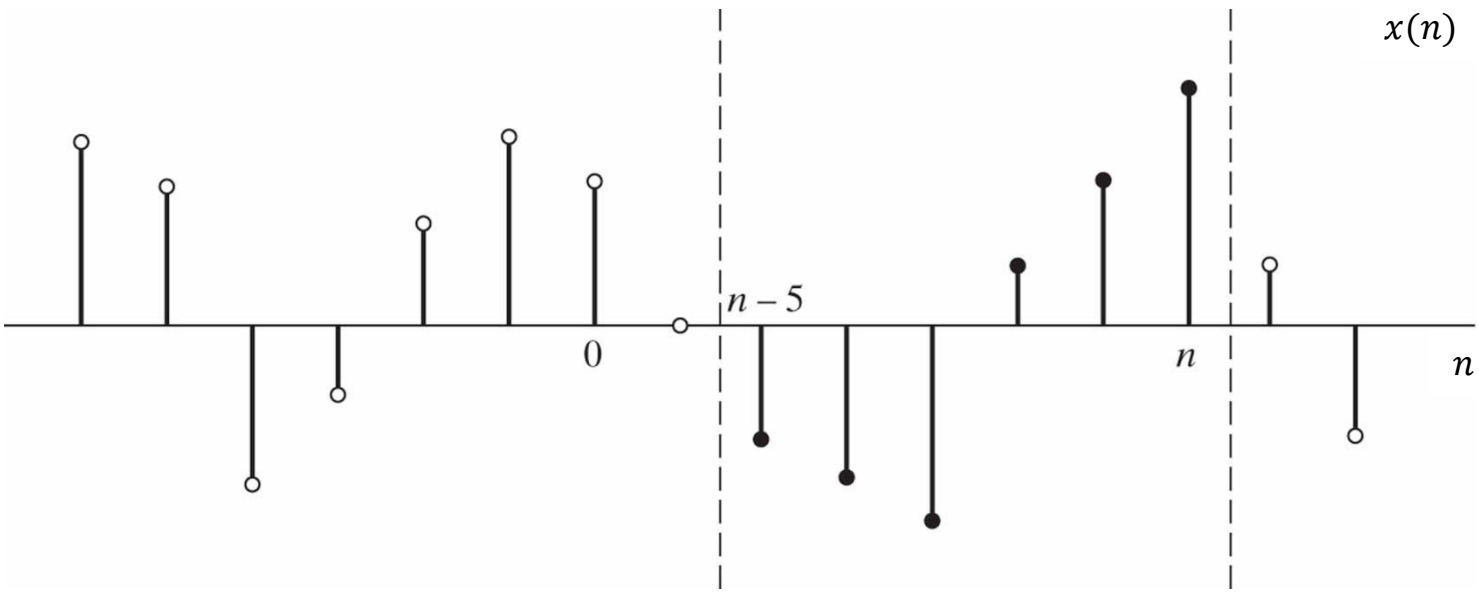
(*median filter*)

$$y(n)=\sum_{k=-\infty}^{n}x(k)=x(n)+x(n-1)+x(n-2)+\cdots$$
$$A: y(n)=\{\cdots,0,3,5,6,\hat{6},7,9,12,12,\cdots\}$$

(*accumulator*)

Example: Moving Average with Window of 6 Samples

$$y(n) = \frac{1}{M_2 - M_1 + 1} \sum_{k=M_1}^{M_2} x(n-k), \quad M_2 \geq M_1$$



Initial Conditions

- **For the accumulator function:** $y(n) = \sum_{k=-\infty}^{n} x(n)$

$$= y(n-1) + x(n)$$

requires 1 initial condition

- ***Example*: Let *x*(*n*) = *nu*(*n*). Find *y*(*n*) assuming *y*(–1) = 0**
- ***Solution*:**

$$y(n) = \sum_{k=-\infty}^{n} x(n) = \sum_{k=-\infty}^{-1} x(k) + \sum_{k=0}^{n} x(k)$$

$$= y(-1) + \sum_{k=0}^{n} k$$

$$= y(-1) + \frac{n(n+1)}{2} = \frac{n(n+1)}{2}$$

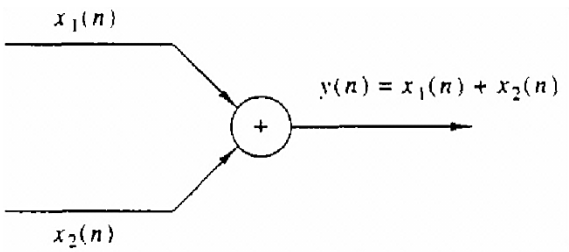
- ***Example*: Find *y*(*n*) assuming *y*(2) = 1**

$$y(n) = \sum_{k=-\infty}^{2} x(k) + \sum_{k=3}^{n} x(k)$$

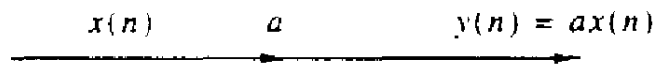
$$= y(2) + \sum_{k=3}^{n} k$$

$$= 1 + \frac{n(n+1)}{2} - 3 = \frac{n(n+1)}{2} - 2$$

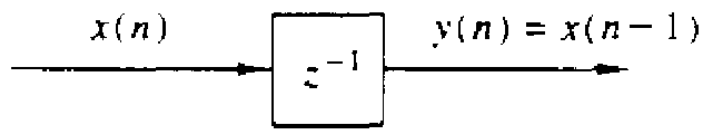
Block Diagram Representation of DT Systems



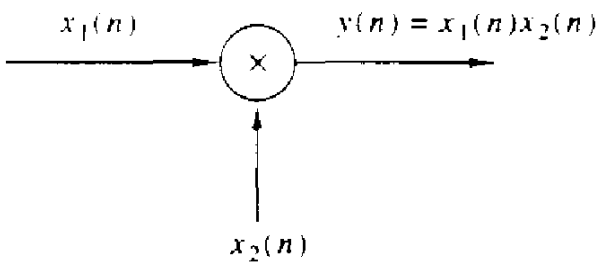
signal adder



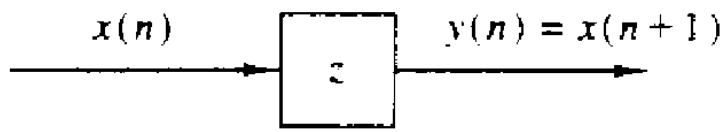
constant multiplier



unit delay



signal multiplier

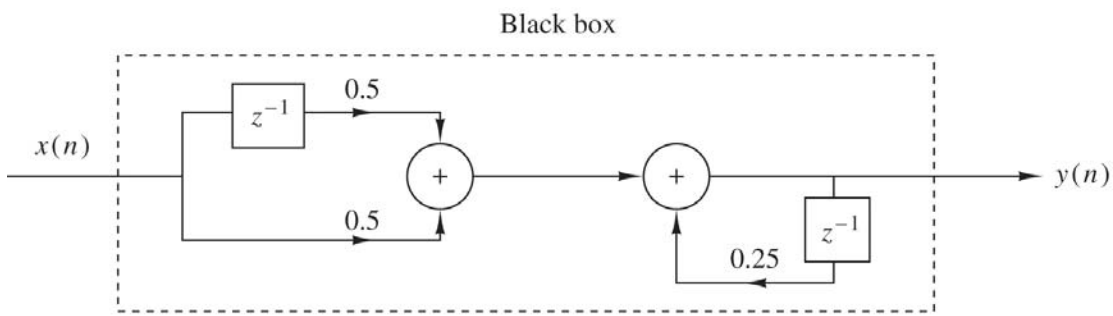


unit advance

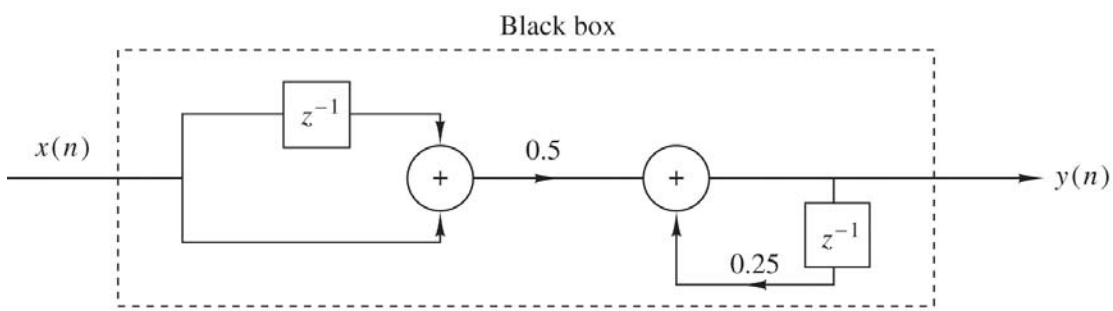
Example

- **What is the input-output relationship that the following block diagram implements?**

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$



(a)



(b)

Time-Varying vs. Time-Invariant Systems

- **Time-invariant (TI): Input-output characteristics do not change with time**
  - Assume system produces an output *y*(*n*) when excited with input signal *x*(*n*)
  - Now assume input is delayed by *k* time units and then applied to system. If the system characteristics do not change, the system should produce the same response as that to *x*(*n*) but delayed by *k* time units.

$$x(n) \xrightarrow{H} y(n)$$

LTI ➔

$$x(n-k) \xrightarrow{H} y(n-k)$$

for every input $x(n)$ and time shift k

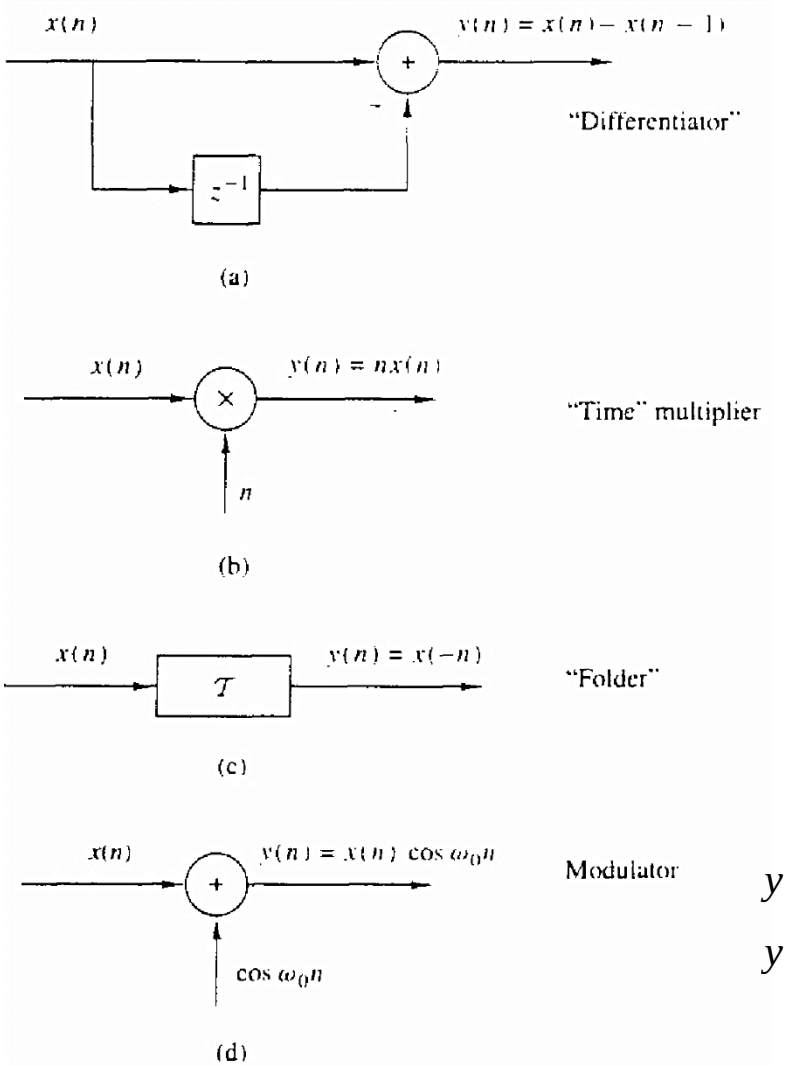
- **In general, delaying *x*(*n*) by *k* time units and applying it to system, we get**

$$y(n,k) = H\left[x(n-k)\right]$$

  - If $y(n,k) = y(n-k)$ for all possible shift values of *k*, system in TI
  - If $y(n,k) \neq y(n-k)$ even for one value of *k*, system is time-varying

Example

- **Determine which of the following systems are TI or time-varying.**



"Differentiator"

$$y(n) = x(n) - x(n-1)$$
$$y(n,k) = x(n-k) - x(n-k-1) = y(n-k)$$

"Time" multiplier

$$y(n) = nx(n)$$
$$y(n,k) = nx(n-k) \neq y(n-k) = (n-k)x(n-k)$$

"Folder"

$$y(n) = x(-n)$$
$$y(n,k) = x(-n-k) \neq y(n-k) = x(-n+k)$$

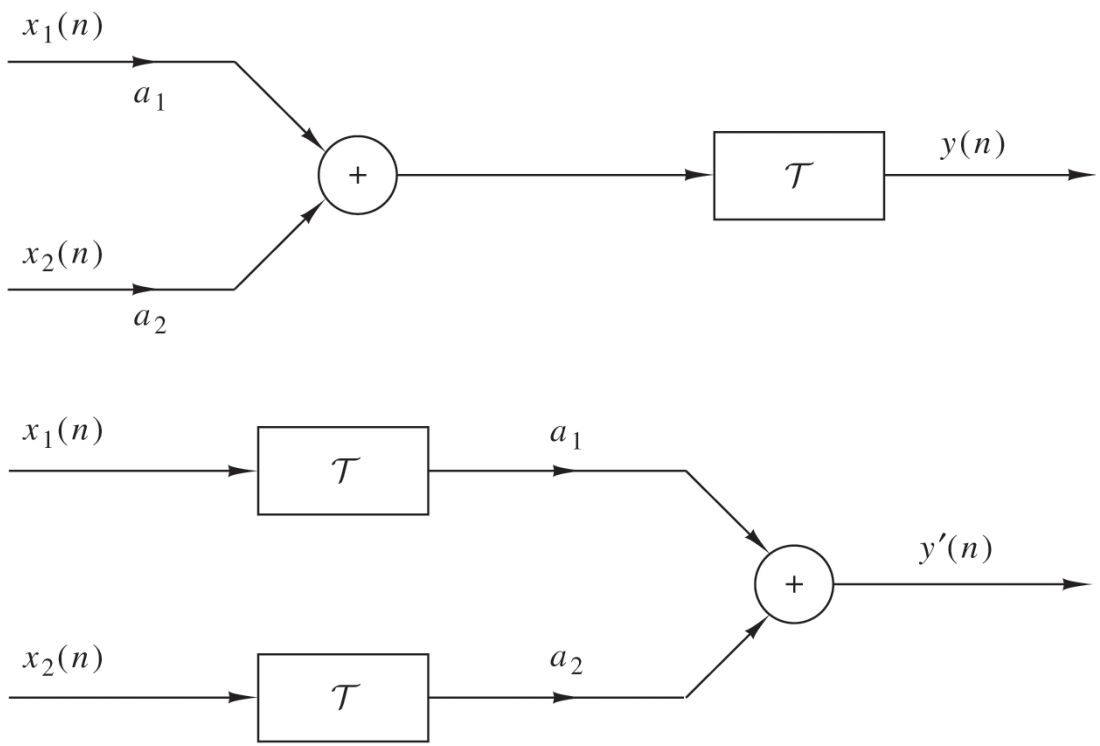
Modulator

$$y(n) = x(n)\cos\omega_0 n$$
$$y(n,k) = x(n-k)\cos\omega_0 n \neq y(n-k) = x(n-k)\cos\omega_0(n-k)$$

Linear vs. Non-Linear Systems

- **A linear system is one that satisfies the superposition principle**

$$H\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 H\left[x_1(n)\right] + a_2 H\left[x_2(n)\right]$$



Example

- **Which of the following systems are linear and which are non-linear.**

1. ***y(n) = n x(n)***

$$y_1(n) = nx_1(n),\ y_2(n) = nx_2(n)$$
$$y_3(n) = n\left[a_1x_1(n) + a_2x_2(n)\right] = a_1nx_1(n) + a_2nx_2(n) = a_1y_1(n) + a_2y_2(n)$$

linear

2. ***y(n) = x(n²)***

$$y_1(n) = x_1\left(n^2\right),\ y_2(n) = x_2\left(n^2\right)$$
$$y_3(n) = a_1x_1\left(n^2\right) + a_2x_2\left(n^2\right) = a_1y_1(n) + a_2y_2(n)$$

linear

3. ***y(n) = x²(n)***

$$y_1(n) = x_1^2(n),\ y_2(n) = x_2^2(n)$$
$$y_3(n) = \left(a_1x_1(n) + a_2x_2(n)\right)^2 \neq a_1y_1(n) + a_2y_2(n) = a_1x_1^2(n) + a_2x_2^2(n)$$

nonlinear

4. ***y(n) = Ax(n) + B***

$$y_1(n) = Ax_1(n) + B,\ y_2(n) = Ax_2(n) + B$$
$$y_3(n) = A\left(a_1x_1(n) + a_2x_2(n)\right) + B \neq a_1y_1(n) + a_2y_2(n) = a_1\left(Ax_1(n) + B\right) + a_2\left(Ax_2(n) + B\right)$$

nonlinear

5. ***y(n) = e^{x(n)}***

$$y_1(n) = e^{x_1(n)},\ y_2(n) = e^{x_2(n)}$$
$$y_3(n) = e^{a_1x_1(n) + a_2x_2(n)} \neq a_1y_1(n) + a_2y_2(n) = a_1e^{x_1(n)} + a_2e^{x_2(n)}$$

nonlinear

Causal vs Non-Causal Systems

- **If the output of system at any time *n* depends only on current and past inputs [i.e. *x*(*n*), x(*n*-1), …] but does not depend on future inputs [i.e. *x*(*n*+1), x(*n*+2), …], it is called *causal*.**
- ***Examples*:**
  - $y(n) = x(n) - x(n-1)$ **causal**
  - $y(n) = \sum_{k=-\infty}^{n} x(k)$ **causal**
  - $y(n) = a\,x(n)$ **causal**
  - $y(n) = x(n) + 3\,x(n+4)$ **non-causal**
  - $y(n) = x(n^2)$ **non-causal**
  - $y(n) = x(2n)$ **non-causal**
  - $y(n) = x(-n)$ **non-causal**

Stable vs. Unstable Systems

- **Bounded-Input, Bounded-Output (BIBO) Stability**
  - A system is BIBO stable if every bounded-input produces a bounded output

- **Hence there exist some finite numbers M_x and M_y such that**

$$|x(n)| \le M_x < \infty, \qquad |y(n)| \le M_y < \infty$$

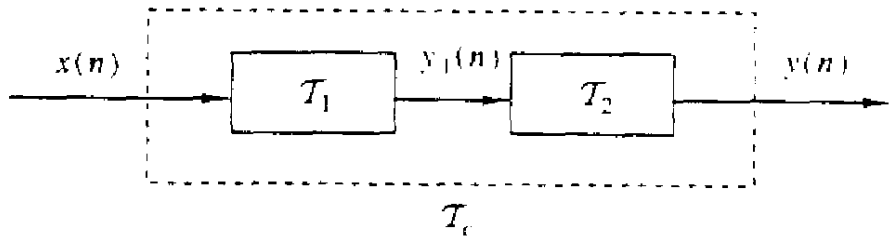
- ***Example*: Is the system with IO relation $y(n) = y^2(n-1) + x(n)$ BIBO stable?**

- ***Solution*: If we excite the system with $x(n) = C\delta(n)$ (which is bounded) where $C > 1$ is a constant, then we obtain the following outputs assuming $y(-1) = 0$:**
  - $y(0) = C$
  - $y(1) = C^2$
  - $y(2) = C^4$
  - …
  - $y(n) = C^{2n}$

  Hence system is not BIBO stable

Interconnection of DT Systems

- **Cascade (series): In general, order matters**
  - For LTI systems T_1 and T_2, order does not matter and combined T_1T_2 system is LTI. Proof?
- **Parallel**



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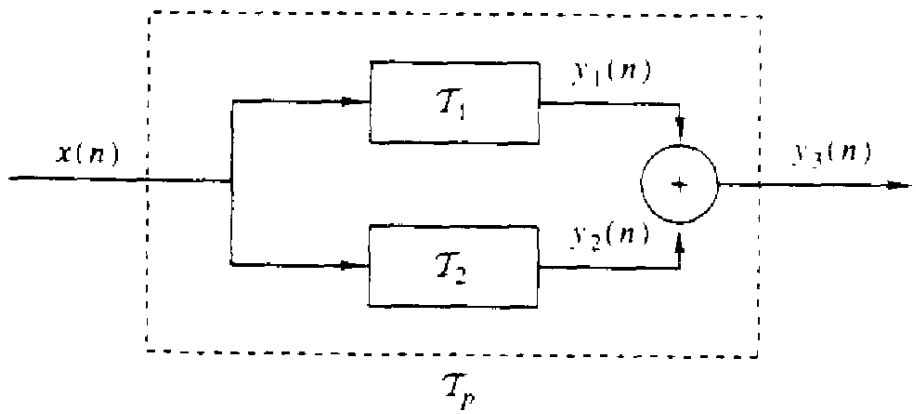


Analysis of Discrete-time LTI Systems

Techniques for Analyzing Linear Systems

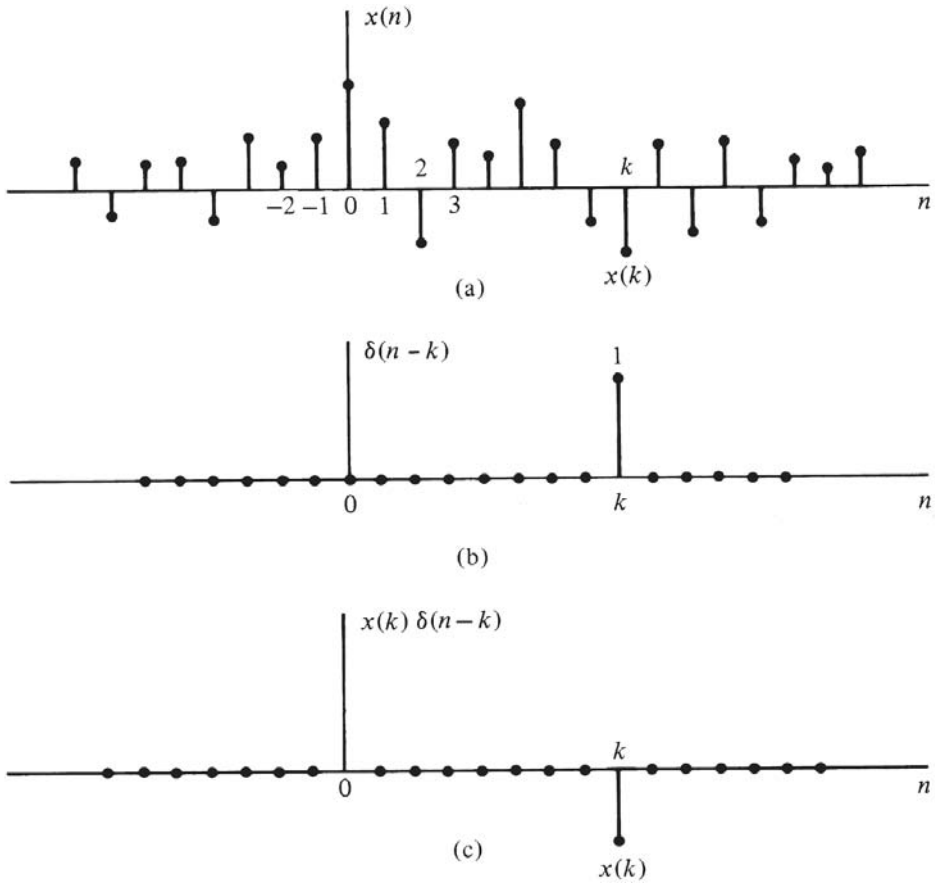
- **<u>Main theme</u>:**
  - Any input signal can be decomposed and represented as a weighted sum of impulses
  - LTI systems are fully characterized by their response to a unit sample (impulse) sequence.
- **Methods to determine behavior of LTI system**
  - Method 1: Directly solve the input-output equation describing the system
  - Method 2: Resolve input into weighted sum of impulses, get response to an impulse, then use linearity and time-invariance properties of system to get response of arbitrary input
- **General form of input-output relationship for an LTI system**

$$\boxed{y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)}$$

  - Called difference equation
  - Function of N past <u>outputs</u>, each weighted by a_k.
  - Function of current and M past <u>inputs</u>, each weighted by b_k.

Resolution of a DT Signal into Impulses

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$



Response of Linear Systems to Arbitrary Inputs: Convolution Sum

- **Denote response $y(n,k)$ of system to the input impulse at $n = k$ by the special symbol $h(n,k)$, for $-\infty < k < \infty$**

$$\boxed{y(n,k) \equiv h(n,k) = \mathcal{T}\left[\delta(n-k)\right]}$$

called impulse response function

time index

location of impulse

- **If the input impulse is scaled by $x(k)$, response is scaled as $x(k)h(n,k)$ as well**
- **If $x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$, then the response is the corresponding sum of weighted outputs:**

$$y(n,k) = \mathcal{T}\left[x(n)\right] = \mathcal{T}\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)\mathcal{T}\left[\delta(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n,k)$$

applies for any linear system

- **The above response is for any linear system (not necessarily LTI)**
  - We only used the linearity or superposition property, but not time-invariance property

If System is Linear + Time-Invariant ….

$$h(n) \equiv \mathcal{T}\left[\delta(n)\right] \xrightarrow{\text{time invariance}} h(n-k) \equiv \mathcal{T}\left[\delta(n-k)\right]$$

$$y(n,k) = \sum_{k=-\infty}^{\infty} x(k)h(n,k) \longrightarrow \boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)}$$

… Convolution Sum

$$y(n) = x(n) * h(n)$$

- **Notes:**
  - LTI system is completely characterized by $h(n)$
  - Otherwise, a time-varying system requires knowledge of an infinite number of impulse response functions $h(n,k)$, one for each delay k.
  - Convolution sum is commutative in x and h

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)}$$

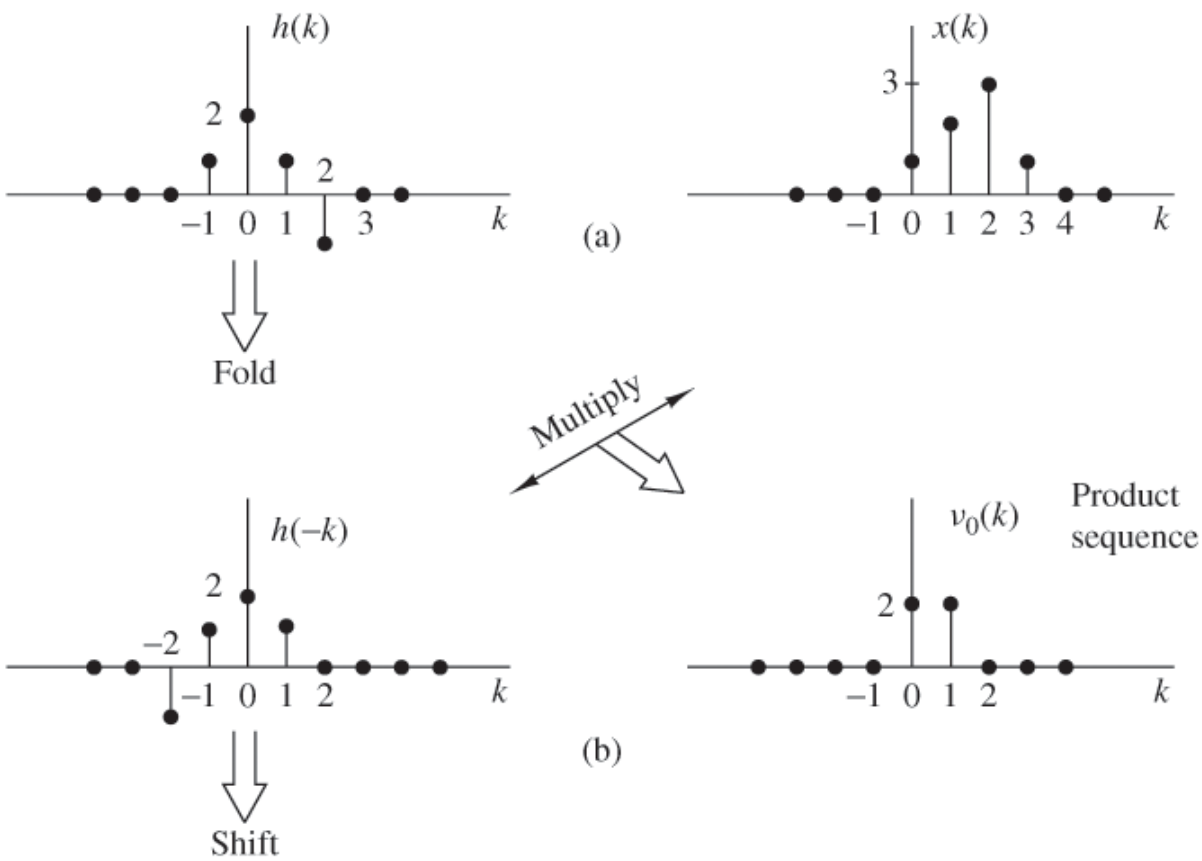
- **How to obtain the response at a particular time instant $n = n_0$?**

$$y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0-k)$$

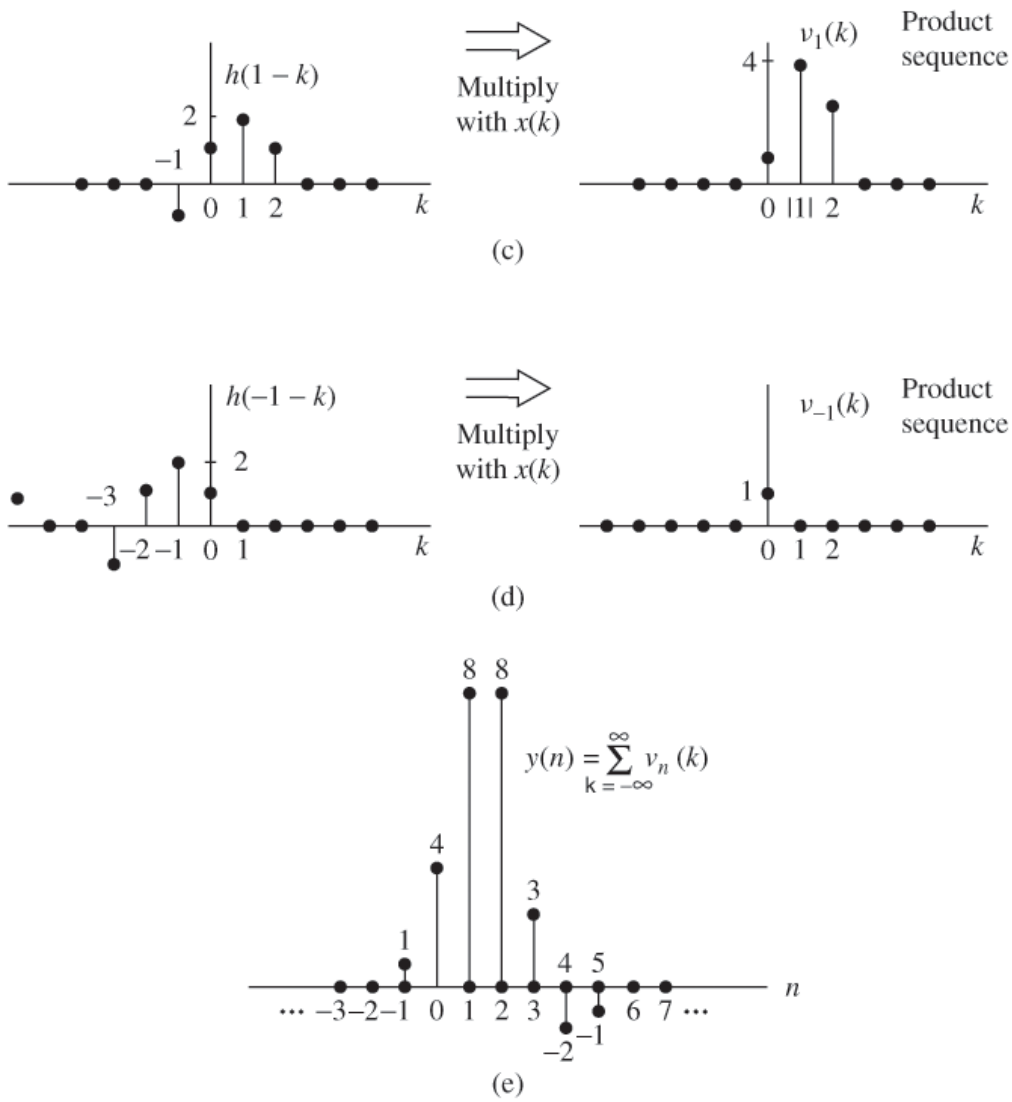
Graphical Computation of Convolution Sum

- ***Example***
  - $h(n) = \{1, \hat{2}, 1, -1\}$
  - $x(n) = \{\hat{1}, 2, 3, 1\}$

$$y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0 - k)$$



Graphical Computation of Convolution Sum (Example cont'd)

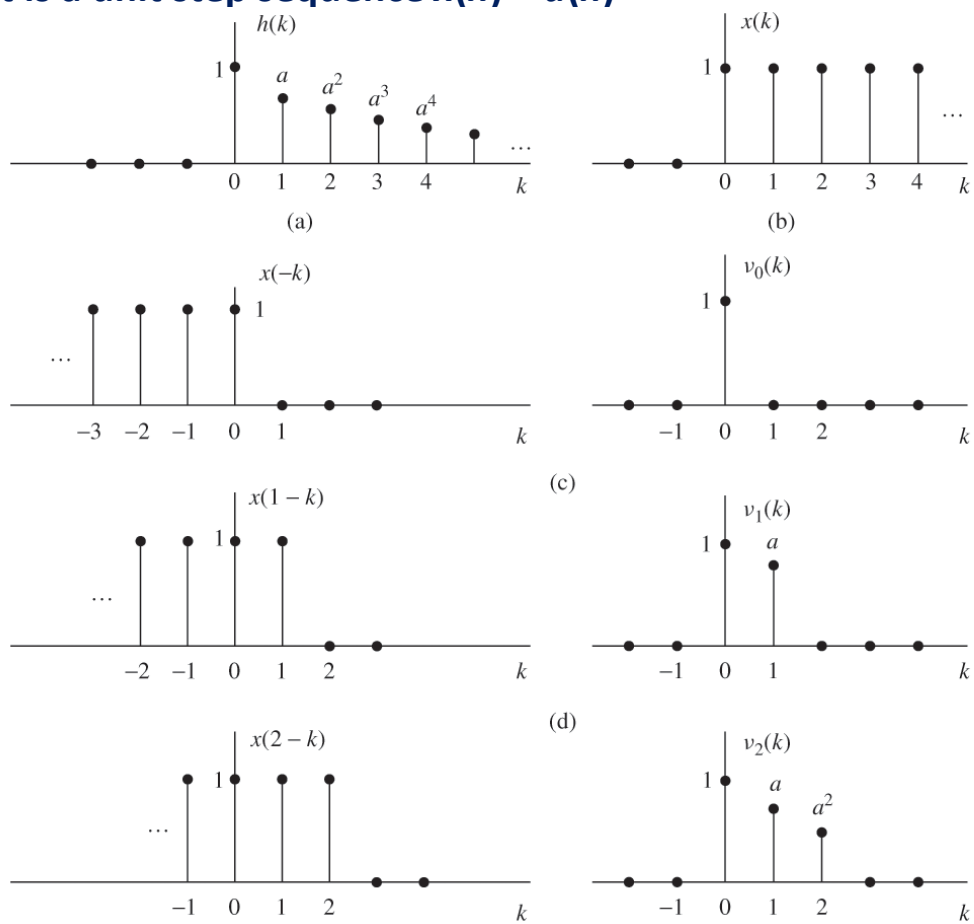


Example

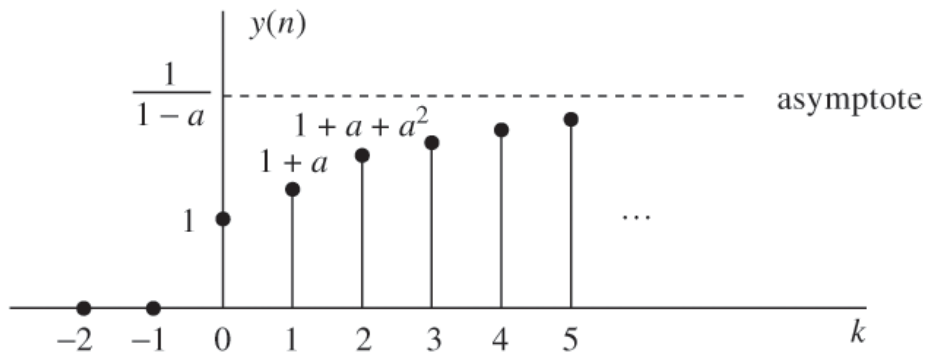
- **Determine *y*(*n*) of an LTI system with impulse response**

$$h(n) = a^n u(n), \quad |a| < 1$$

when the input is a unit step sequence *x*(*n*) = *u*(*n*)



Example (cont'd)



(f)

$$y(n) = 1 + a + a^2 + \cdots + a^2$$

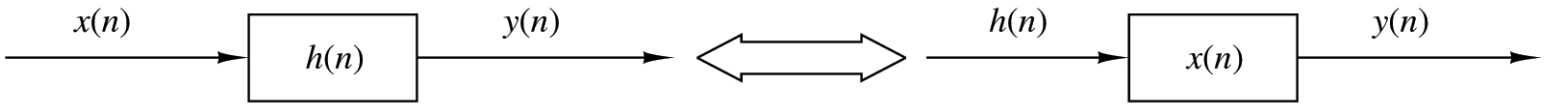
$$= \frac{1 - a^{n+1}}{1 - a} \qquad \text{for} \quad n \geq 0$$

$$y(n) = 0 \qquad \text{for} \quad n < 0$$

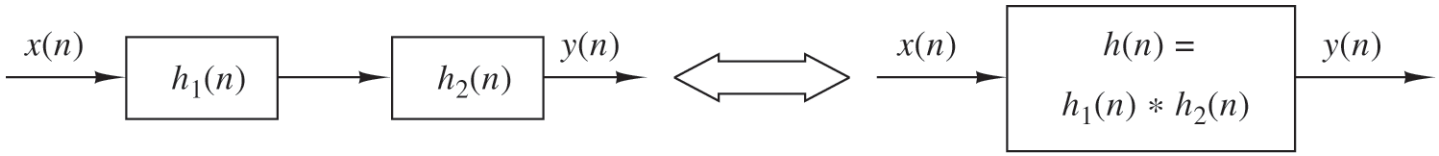
Properties of Convolution

$$y(n) = x(n) * \delta(n) = x(n)$$
$$x(n) * \delta(n-k) = y(n-k) = x(n-k)$$

- **Commutativity:** $y(n) = x(n) * h(n) = h(n) * x(n)$



- **Associativity:** $\left(x(n) * h_1(n)\right) * h_2(n) = x(n) * \left(h_1(n) * h_2(n)\right)$



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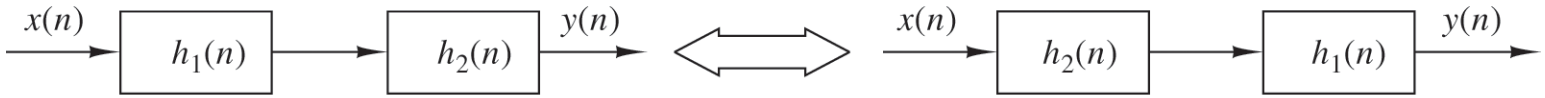


(b)

Example

- **Determine impulse response for the cascade of 2 LTI systems with**

$$h_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$h_2(n) = \left(\frac{1}{4}\right)^n u(n)$$

- ***Solution*:** $h(n) = h_1(n) * h_2(n)$

$$h(n) = \sum_{k=-\infty}^{\infty} h_1(k) h_2(n-k)$$

$$h_1(k) h_2(n-k) = \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k} = \begin{cases} \neq 0 & k \geq 0 \text{ or } n \geq k \geq 0 \\ 0 & n < 0 \end{cases}$$

- **Hence for $n \geq k \geq 0$, we have**

$$h(n) = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k}$$

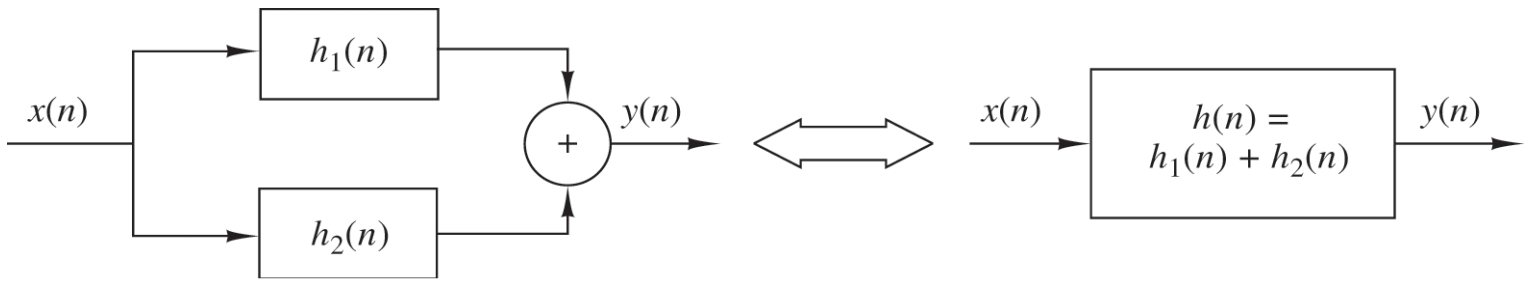
$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 2^k$$

$$= \left(\frac{1}{2}\right)^n \left[2 - \left(\frac{1}{2}\right)^n\right], \quad n \geq 0$$

Properties of Convolution (cont'd)

- **Distributive law**

$$x(n)*\left(h_1(n)+h_2(n)\right)=x(n)*h_1(n)+x(n)*h_2(n)$$



Causal LTI Systems

- **For LTI systems, causality can be translated into a condition on *h*(*n*)**

$$y(n_0) = \sum_{k=-\infty}^{\infty} h(k)\,x(n_0-k)$$

$$= \underbrace{\sum_{k=0}^{\infty} h(k)\,x(n_0-k)}_{\text{current \& past values}} + \underbrace{\sum_{k=-\infty}^{-1} h(k)\,x(n_0-k)}_{\text{future values}}$$

Causal → 0

- **Causal LTI → *h*(*n*) = 0 for *n* < 0**
- **Two equivalent forms:**

$$\boxed{\begin{aligned} y(n) &= \sum_{k=0}^{\infty} h(k)\,x(n-k) \\ &= \sum_{k=-\infty}^{n} x(k)\,h(n-k) \end{aligned}}$$

- **If the input is a *causal sequence*, i.e., *x*(*n*) = 0 for *n* < 0, then convolution becomes:**

$$\boxed{\begin{aligned} y(n) &= \sum_{k=0}^{n} h(k)\,x(n-k) \\ &= \sum_{k=0}^{n} x(k)\,h(n-k) \end{aligned}}$$

Example

- **Determine the unit step response of the LTI system with impulse response**

$$h(n) = a^n u(n), \quad |a| < 1$$

- ***Solution*: Since *h*(*n*) is causal, and *x*(*n*) = *u*(*n*) is causal, then**

$$y(n) = \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad \text{for } n \geq 0$$

$$y(n) = 0 \qquad \text{for } n < 0$$

Stability of LTI Systems

- **An LTI system is BIBO stable if and only if h[n] is absolutely summable:**

$$\boxed{S_h = \sum_{k=-\infty}^{\infty} \left| h(k) \right| < \infty}$$

Finite-Duration and Infinite-Duration Impulse Response

- **Finite-Impulse Response (FIR) of order M:**

$$h(n) = 0, \quad n < 0 \quad \text{and} \quad n \geq M$$

$\Rightarrow$ **convolution sum becomes**

$$y(n) = \sum_{k=0}^{M-1} h(k)\,x(n-k)$$

- **Hence FIR system operates on a window of only the most recent M values of $x(n)$**

- **Infinite-Impulse Response (IIR): Has infinite duration impulse response**

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k) \qquad \text{(causality assumed)}$$

system has infinite memory

DT Systems Described by Difference Equations

Recursive and Non-recursive DT Systems

- **Can a system described by the input-output relationship below be implemented?**

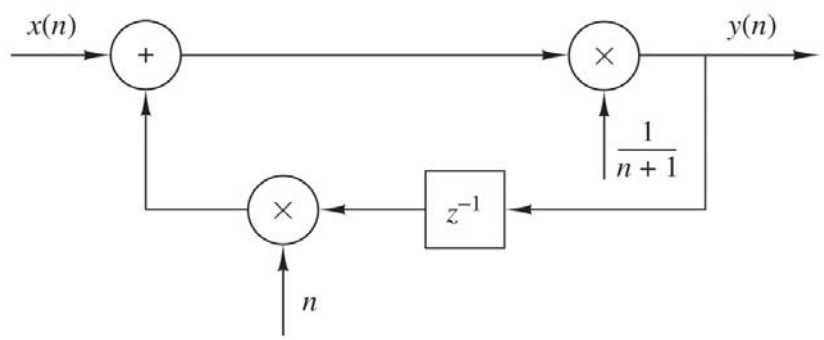
$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

- **Clearly, for FIR case, it is possible**
- **What bout IIR case?**
  - If we can express the current output in terms of current and past inputs, as well as past outputs, then such a system can be physically realized
  - This class of LTI IIR systems that can be expressed in terms of **difference equations** is physically realizable
- ***Example*: Cumulative average system**

$$y(n) = \frac{1}{n+1}\sum_{k=0}^{n} x(k), \quad n = 0,1,2,\cdots$$

  - Memory grows with n
  - We can express as a recursion

$$y(n) = \frac{n}{n+1}y(n-1) + \frac{1}{n+1}x(n)$$



Example: Square-Root Algorithm

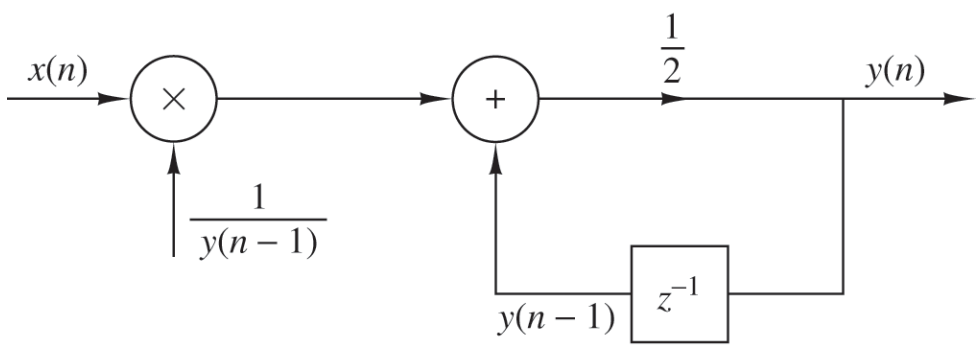
- **Calculators implement square-root of $A > 0$ using the iterative algorithm**

$$s_n = \frac{1}{2}\left(s_{n-1} + \frac{A}{s_{n-1}}\right), \quad A > 0, \quad n = 0,1,2,\cdots$$

- **The following recursive system when excited with *x*(*n*) = *Au*(*n*) with appropriate initial condition, the response will converge toward $\sqrt{A}$ as n increases**

$$y(n) = \frac{1}{2}\left(y(n-1) + \frac{A}{y(n-1)}\right)$$

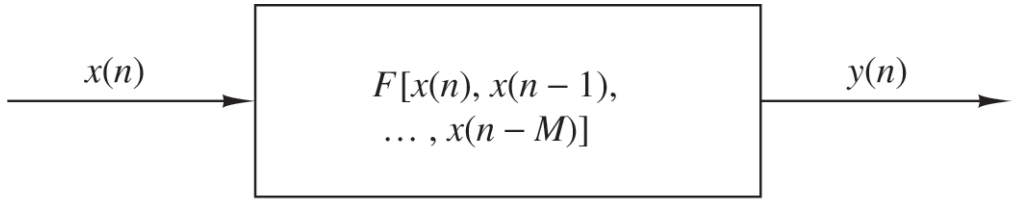
  – We do not need to specify an exact initial condition y(-1). A rough estimate is enough



Recursive and Non-recursive DT Systems

- **Basic form of causal and realizable non-recursive and recursive systems**

Non-recursive



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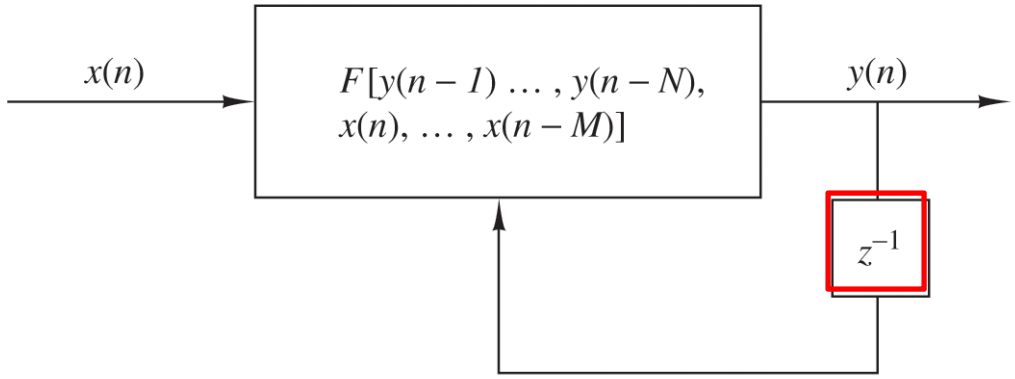
Recursive



(b)

LTI Systems with Constant-Coefficient Difference Equations

- **Systems described by constant-coefficient difference equations are LTI systems:**

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

- **Principle of linearity results in:**
  - Total response is the sum of the zero-input and zero-state responses:

$$y(n) = y_{zi}(n) + y_{zs}(n)$$

  - Zero-input response: Solve for $y(n)$ when $x(n) = 0$
  - Zero-state response: Solve for $y(n)$, $n \geq 0$, when $y(-1) = 0$.

Solution of Linear Constant-Coefficient Difference Equations

- **Goal is to determine output $y(n)$, $n \geq 1$, of system given a specific input $x(n)$, $n \geq 0$ and a set of initial conditions**

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

- **Solution composed of homogeneous solution $y_h(n)$ and particular solution $y_p(n)$**

$$y(n) = y_h(n) + y_p(n)$$

- **$y_h(n)$: solution to the homogeneous difference equation**

$$\sum_{k=0}^{N} a_k y(n-k) = 0, \quad a_0 = 1$$

- **Form characteristic polynomial: Solution is in the form of an exponential, $y_h(n)=\lambda^n$**

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0$$

  - Solution has N roots, $\lambda_1 \ldots \lambda_N$: $y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \cdots + C_N\lambda_N^n$

- **Coefficients are determined from the initial conditions of the system**
  - Result gives zero-input response $y_{zi}(n)$

Solution of Linear Constant-Coefficient Difference Equations

- **Particular solution: Solution $y_p(n)$ required to satisfy DE for the specific input $x(n)$**
- **$y_p(n)$ is any solution satisfying**

$$\sum_{k=1}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k), \quad a_0 = 1$$

- **We assume $y_p(n)$ has a form that depends on the form of the input $x(n)$**

Input Signal, $x(n)$	Particular Solution, $y_p(n)$
A (constant)	K
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \cdots + K_M$
$A^n n^M$	$A^n(K_0 n^M + K_1 n^{M-1} + \cdots + K_M)$
$\left\{ \begin{matrix} A\cos\omega_0 n \\ A\sin\omega_0 n \end{matrix} \right\}$	$K_1 \cos\omega_0 n + K_2 \sin\omega_0 n$

- **$y_p(n)$ is the limiting value of the zero-state response: (steady-state response)**

$$y_p(n) = \lim_{n\to\infty} y_{zs}(n)$$

Example

- **Determine solution of the system described by the first-order DE when *x*(*n*) = *u*(*n*):**

$$y(n)+a_1 y(n-1)=x(n), \quad |a_1|<1$$

- ***Solution*:**
  - Homogeneous solution: set $x(n) = 0$. The assumed solution has the form $y_h(n)=\lambda^n$

$$\lambda^n + a_1\lambda^{n-1} = \lambda^{n-1}(\lambda + a_1) = 0$$
$$\Rightarrow \lambda = -a_1$$

$$y_h(n) = C\lambda^n = C(-a_1)^n$$

  - To get zero-input response, find coefficient C. Use initial conditions of the system $y(-1)$
  - When $x(n) = 0$, we have $y(0) = a_1 y(-1)$.
  - Also, $y_h(0) = C$
  - Therefore, $y_{zi}(n) = (-a_1)^{n+1} y(-1)$

  - Particular solution when $x(n) = u(n)$: Assume $y_p(n)$ also has the form $y_p(n) = Ku(n)$
  - Substitute in DE: $Ku(n) + a_1 Ku(n-1) = u(n)$
    - Evaluate for any $n \geq 1$ where none of the terms vanish: $K + a_1 K = 1 \Rightarrow K = \dfrac{1}{1+a_1}$

$$\Rightarrow \quad y_p(n) = \frac{1}{1+a_1} u(n)$$

Example (cont'd)

- **Total solution:**

$$y(n) = C(-a_1)^n + \frac{1}{1+a_1}, \quad n \geq 0$$

- **The constant is determined to satisfy the initial condition $y(-1)$. For example, to determine the zero-state response, set $y(-1) = 0$ and evaluate DE at $n = 0$.**
- **To get C for any $y(-1)$, we evaluate DE at $n = 0$**

$$y(n) + a_1 y(n-1) = u(n)$$
$$\Rightarrow y(0) + a_1 y(-1) = u(0)$$
$$\Rightarrow y(0) = 1 - a_1 y(-1)$$

- **On the other hand:**

$$y(0) = C(-a_1)^0 + \frac{1}{1+a_1} = C + \frac{1}{1+a_1}$$

- **Therefore**

$$C = -a_1 y(-1) + \frac{a_1}{1+a_1}$$

- **Substitute back in total solution:**

$$y(n) = (-a_1)^{n+1} y(-1) + \frac{1-(-a_1)^{n+1}}{1+a_1}, \quad n \geq 0$$
$$= y_{zi}(n) + y_{zs}(n)$$

Example

- **Determine the particular solution of the DE when $x(n) = \mathbf{2^n u(n)}$:**

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$$

- ***Solution*: $y_p(n) = K\mathbf{2^n u(n)}$**

$$K2^n u(n) = \frac{5}{6}K2^{n-1}u(n-1) - \frac{1}{6}K2^{n-2}u(n-2) + 2^n u(n)$$

- **To determine K, evaluate above equation for any $n \geq 2$**

$$K2^2 = \frac{5}{6}K2^1 - \frac{1}{6}K2^0 + 2^2 \quad \Rightarrow \quad K = \frac{8}{5}$$

$$\Rightarrow \quad y_p(n) = \frac{8}{5}2^n u(n)$$

Example

- **Determine the response $y(n)$ for $n \geq 0$ of the system described by the second-order DE when $x(n) = 4^n u(n)$:**

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

- ***Solution*: Homogeneous part has the form $y_h(n) = \lambda^n$**

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = \lambda^{n-2}\left(\lambda^2 - 3\lambda - 4\right) = 0$$

$$\Rightarrow \lambda_1 = -1, \lambda_2 = 4$$

$$\boxed{y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n = C_1(-1)^n + C_2(4)^n}$$

  - To get zero-input solution, use initial conditions $y(-1)$ and $y(-2)$ of the system:
    - $y(0) = 3y(-1) + 4y(-2) = C_1 + C_2$
    - $y(1) = 3y(0) + 4y(-1) = 13y(-1) + 12y(-2) = -C_1 + 4C_2$
    - $\therefore C_1 = -\frac{1}{5}y(-1) + \frac{4}{5}y(-2), \qquad C_2 = \frac{16}{5}y(-1) + \frac{16}{5}y(-2)$

$$\boxed{y_{zi}(n) = \left(-\frac{1}{5}y(-1) + \frac{4}{5}y(-2)\right)(-1)^n + \left(\frac{16}{5}y(-1) + \frac{16}{5}y(-2)\right)(4)^n, \quad n \geq 0}$$

Example (cont'd)

- Particular solution when $x(n) = 4^n u(n)$: Assume $y_p(n)$ also has the form $y_p(n) = K4^n u(n)$
- **Observe that this particular solution is already included in the homogeneous solution**
- **We select the particular solution to be linearly independent of the homogeneous part**

$$\Rightarrow \quad y_p(n) = Kn4^n u(n)$$

- **Substitute back in DE and evaluate for $n \geq 2$, we get $K = \frac{6}{5}$**

$$\boxed{y_p(n) = \frac{6}{5} n4^n u(n)}$$

- **Total solution:**

$$\boxed{\Rightarrow \quad y(n) = C_1(-1)^n + C_2(4)^n + \frac{6}{5} n4^n u(n)}$$

- **Next, determine the zero-state response ($y(-1) = y(-2) = 0$). From DE, we have**
  - $y(0) = 3y(-1) + 4y(-2) + 1 = 1 = C_1 + C_2$
  - $y(1) = 3y(0) + 4y(-1) + 6 = 13y(-1) + 12y(-2) + 9 = 9 = -C_1 + 4C_2 + \frac{24}{5}$
  - Get $C_1 = -\frac{1}{25}$ and $C_2 = \frac{26}{25}$

$$\boxed{\Rightarrow \quad y_{zs}(n) = -\frac{1}{25}(-1)^n + \frac{26}{25}(4)^n + \frac{6}{5} n4^n u(n)}$$

Impulse Response of LTI Recursive Systems

- **Step response: Response to $x(n) = u(n)$**
- **Impulse response $h(n)$: Response to $x(n) = \delta(n)$**
- **In case of a recursive system, $h(n)$ is the *zero-state response* to $x(n) = \delta(n)$ and system is initially at rest**
- ***Example*: $y(n) = ay(n-1) + x(n)$**

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k), \quad n \geq 0$$

$$y_{zs}(n) = \sum_{k=0}^{n} a^k \delta(n-k), \quad n \geq 0$$

$$= a^n, \quad n \geq 0$$

$$\Rightarrow h(n) = a^n u(n)$$

- **General case:**

$$y_{zs}(n) = \sum_{k=0}^{n} h(k) x(n-k), \quad n \geq 0$$

$$= \sum_{k=0}^{n} h(k) \delta(n-k), \quad n \geq 0$$

$$= h(n)$$

Example

- **Determine the impulse response of**

$$y(n)-3y(n-1)-4y(n-2)=x(n)+2x(n-1)$$

- ***Solution*: We already know the homogeneous solution to be**

$$\boxed{y_h(n)=C_1\lambda_1^n+C_2\lambda_2^n=C_1(-1)^n+C_2(4)^n, \quad n\geq 0}$$

  - Need to get the constants C_1 and C_2 when $x(n)=\delta(n)$ and $y(-1)=y(-2)=0$
  - $y(0)=3y(-1)+4y(-2)+\delta(0)=1=C_1+C_2$
  - $y(1)=3y(0)+4y(-1)+2\delta(0)=3+2=5=-C_1+4C_2$
  - Get $C_1=-\frac{1}{5}$ and $C_2=\frac{6}{5}$
  - Therefore

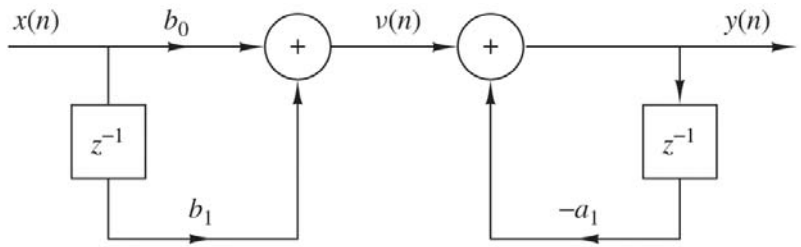
$$\boxed{h(n)=\left[-\frac{1}{5}(-1)^n+\frac{6}{5}(4)^n\right]u(n)}$$

Implementation of DT Systems

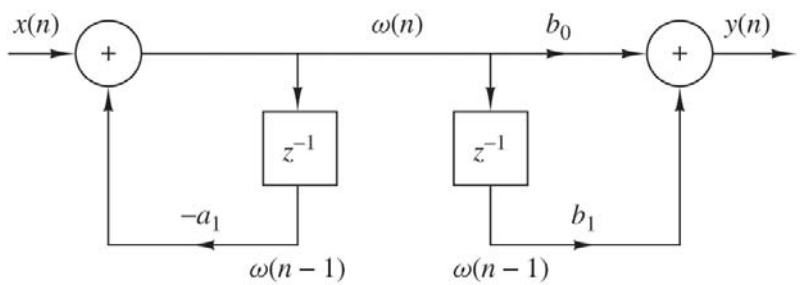
Direct Form Structures

- **Consider the 1st order system** $y(n) = -a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$

Direct Form I

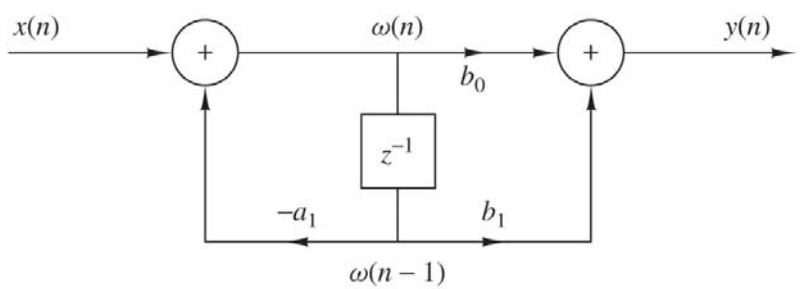


(a)



(b)

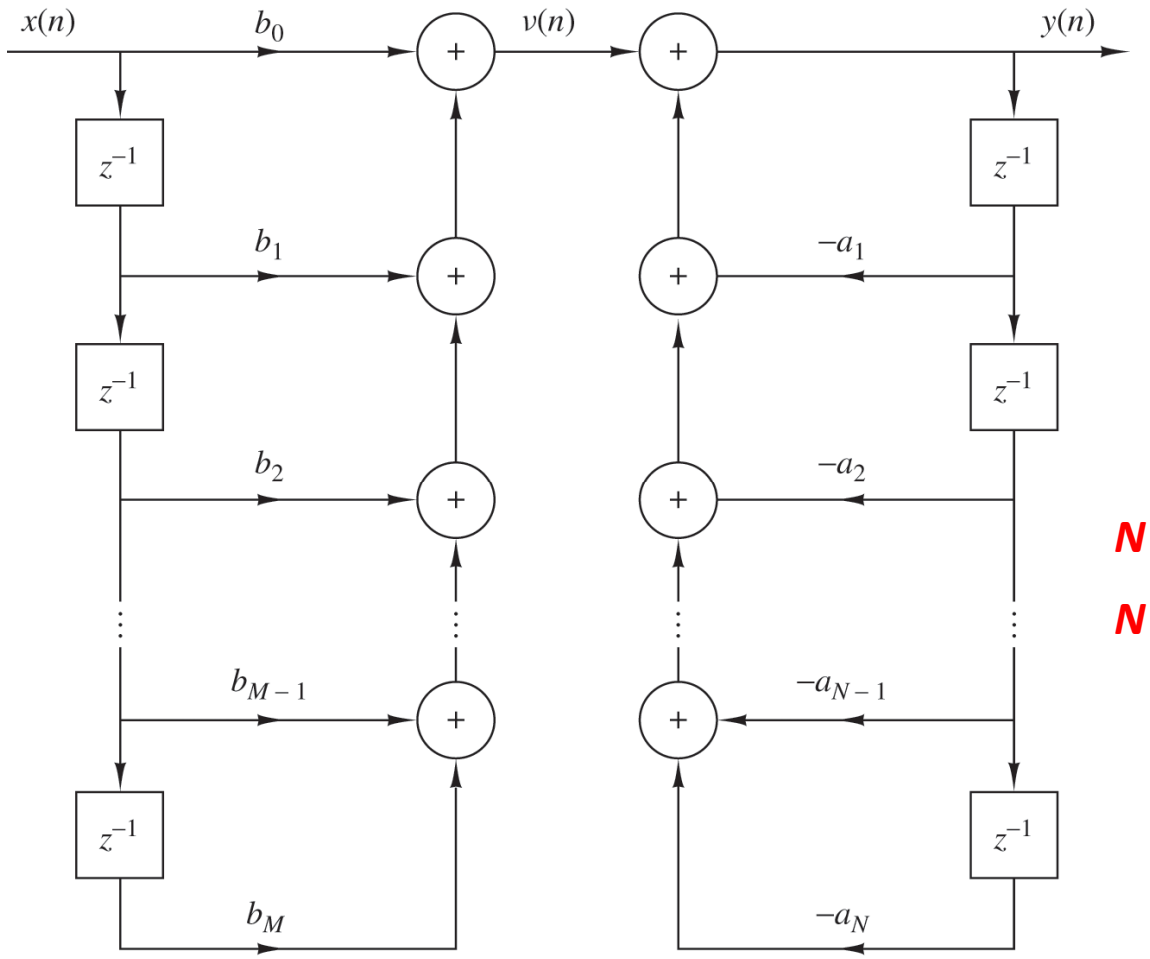
Direct Form II



(c)

Direct Form I Structure

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

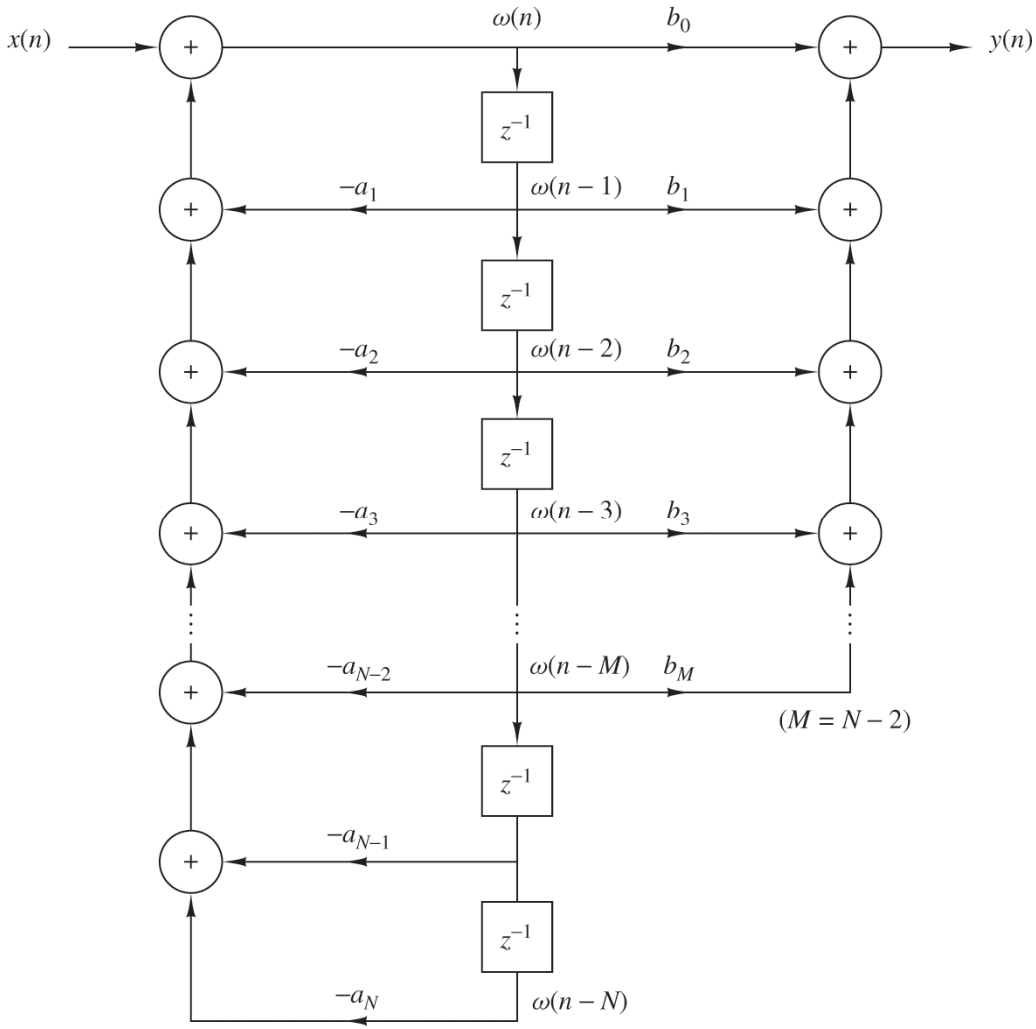


***N* + *M* delays**

***N* + *M* + 1 multiplications**

Direct Form II Structure

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$



N delays if $N \geq M$

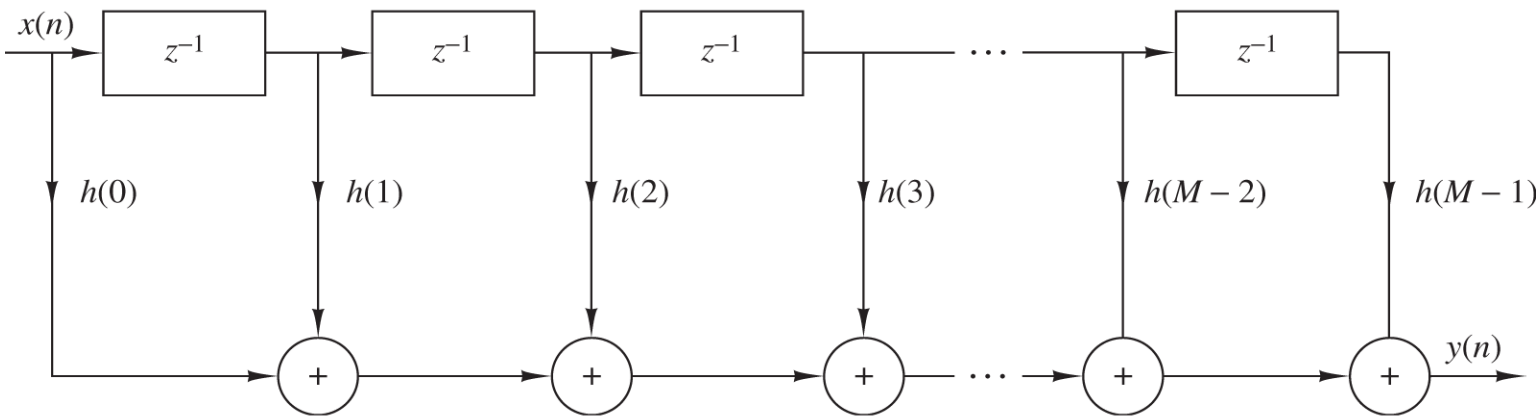
$N + M + 1$ multiplications

(*called canonic form*)

Direct Form FIR Structure

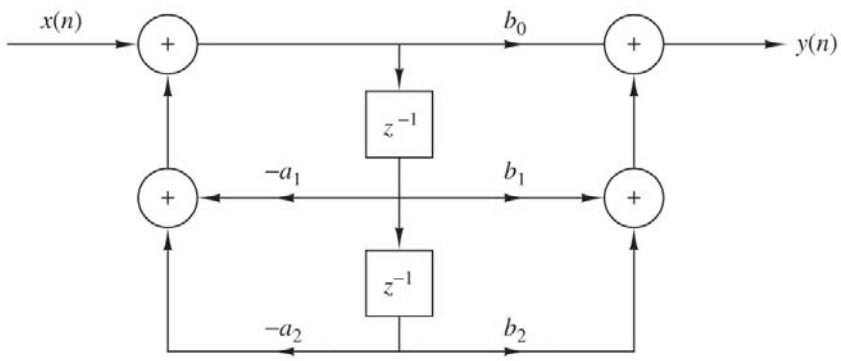
$$y(n) = \sum_{k=0}^{M} b_k x(n-k)$$

$$h(k) = \begin{cases} b_k & 0 \le k \le M \\ 0 & \text{otherwise} \end{cases}$$



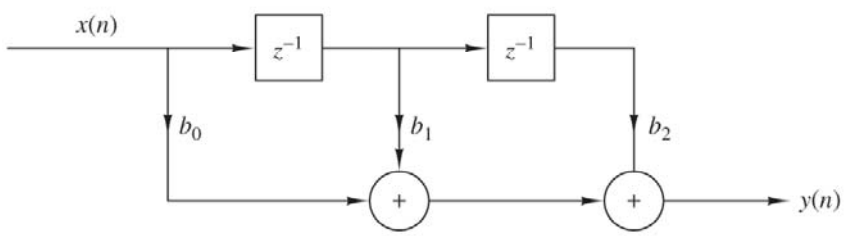
Example

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$



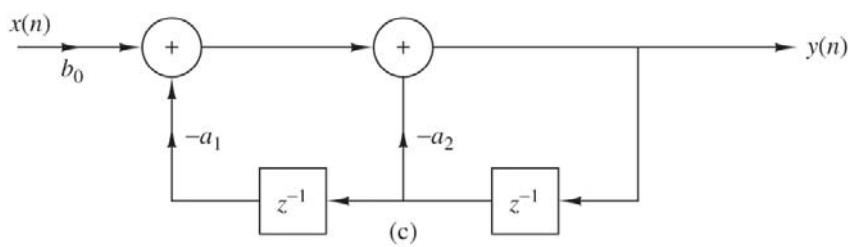
(a)

Direct Form II



(b)

FIR when $a_1 = a_2 = 0$



(c)

Purely recursive when $b_1 = b_2 = 0$

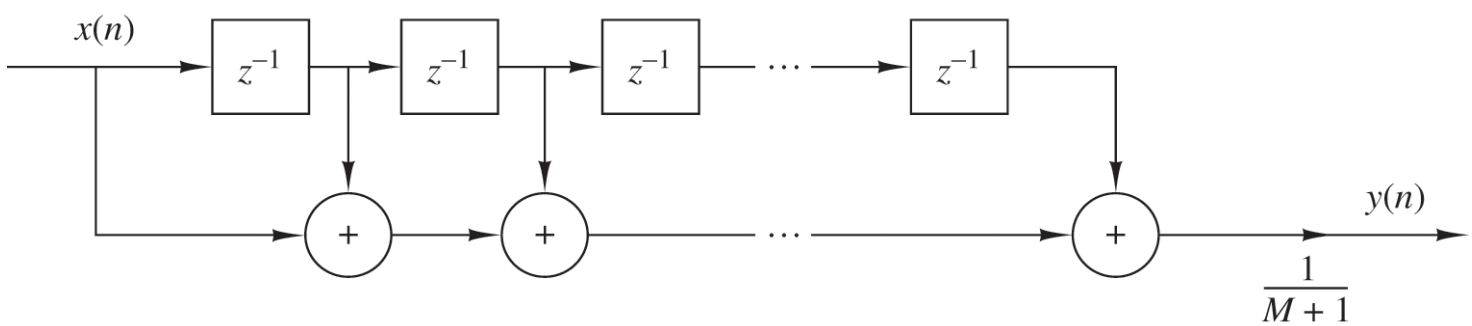
Recursive and Non-recursive Realizations of FIR Systems

- ***Example*: Moving average**

$$y(n) = \frac{1}{M+1} \sum_{k=0}^{M} x(n-k)$$

$$h(n) = \begin{cases} \dfrac{1}{M+1} & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases}$$

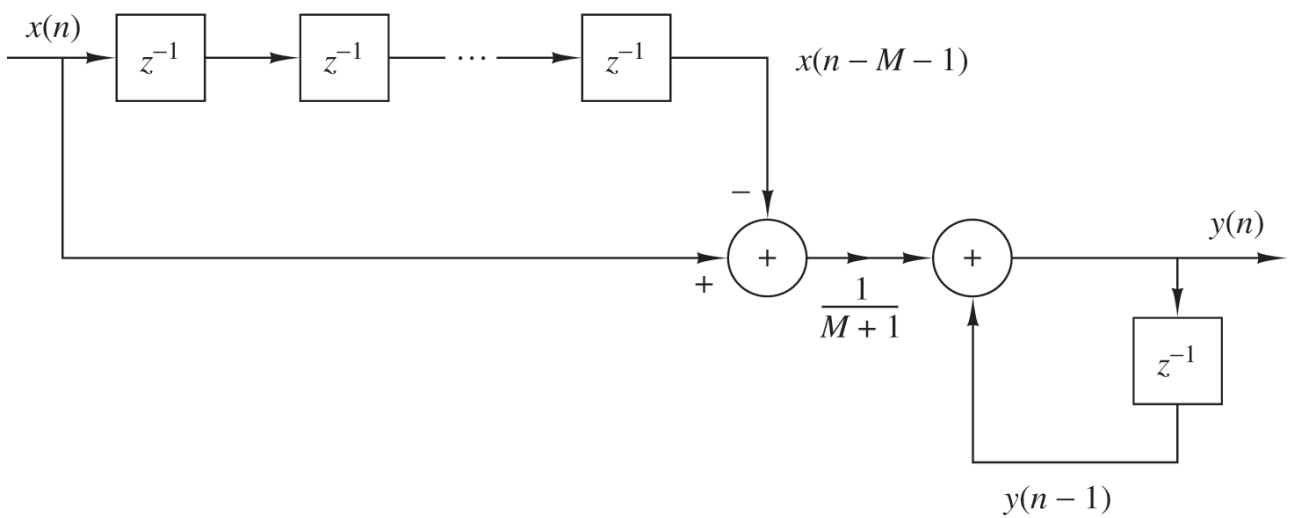
- **Non-recursive realization**



Moving Average (cont'd)

- **We can express moving average system as**

$$y(n) = \frac{1}{M+1}\sum_{k=0}^{M} x(n-1-k) + \frac{1}{M+1}\left[x(n) - x(n-1-M)\right]$$
$$= y(n-1) + \frac{1}{M+1}\left[x(n) - x(n-1-M)\right]$$

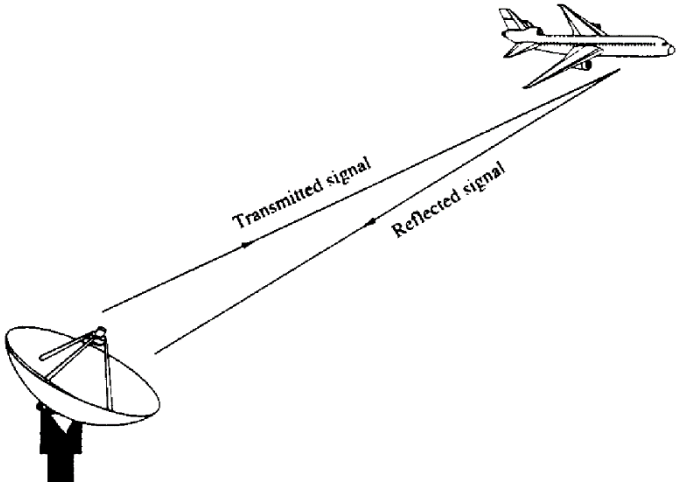


- **Summary: FIR and IIR type of LTI systems, while recursive/non-recursive are descriptions of the structures for realizing or implementing the system**

Correlation of DT Signals

Correlation

- **Measures degree to which two signals are similar**
- **Applications:**
  - Radar, sonar, digital comm., geology, etc.



- ***Example*: In radar**
  - $x(n)$: transmitted signal
  - $y(n) = ax(n - D) + w(n)$ is received noisy version of $x(n)$, that is delayed and attenuated in case there is a target
  - $y(n)$ is just noise in case there is no target
  - Using reference signal $x(n)$ and received signal $y(n)$
    - Compare $x(n)$ and $y(n)$ to determine if a target is present
    - If so, determine the time delay D and the distance of the target

Crosscorrelation and Autocorrelation Sequences

- **Consider two real signal sequences *x*(*n*) and *y*(*n*), with finite energy**
- **Crosscorrelation *x*(*n*) and *y*(*n*) is a sequence $r_{xy}(l)$:**

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-l), \quad l = 0, \pm 1, \pm 2, \ldots$$

$$= \sum_{n=-\infty}^{\infty} x(n+l)\,y(n), \quad l = 0, \pm 1, \pm 2, \ldots$$

  - *l*: shift or lag parameter
  - Subscripts *xy*: sequence *x*(*n*) is not shifted, *y*(*n*) is shifted left or right by *l* steps. Equivalently, *x*(*n*) is shifted right or left by *l* steps relative to *y*(*n*), while leaving *y*(*n*) not shifted

- **Reversing the order of indices, we get**

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)\,x(n-l)$$

$$= \sum_{n=-\infty}^{\infty} y(n+l)\,x(n)$$

$$\therefore \quad r_{xy}(l) = r_{yx}(-l)$$

Example

$$x(n) = \{\cdots, 0, 0, 2, -1, 3, 7, \hat{1}, 2, -3, 0, 0, \cdots\}$$
$$y(n) = \{\cdots, 0, 0, 1, -1, 2, -2, \hat{4}, 1, -2, 5, 0, 0, \cdots\}$$

$$r_{xy}(0) = \sum\{\cdots, 0, 0, 2, 1, 6, -14, 4, 2, 6, 0, 0, \cdots\} = 7$$
$$r_{xy}(1) = 13$$
$$r_{xy}(2) = -18$$
$$r_{xy}(3) = 16$$
$$\cdots$$
$$r_{xy}(l) = 0, \quad l \geq 7$$

Relation to Convolution

- **Crosscorrelation is similar to convolution except for the folding operation**
- **Hence:**

$$r_{xy}(l) = x(l) * y(-l)$$

- **Note that absence of folding makes crosscorrelation a non-commutative operation**
- **Special case when *x*(*n*) = *y*(*n*): Operation called autocorrelation**

$$\begin{aligned} r_{xx}(l) &= \sum_{n=-\infty}^{\infty} x(n)x(n-l), \quad l = 0, \pm 1, \pm 2, \ldots \\ &= \sum_{n=-\infty}^{\infty} x(n+l)x(n), \quad l = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

- **In case *x*(*n*), *y*(*n*) are causal sequences of finite duration *N*:**

$$r_{xy}(l) = \sum_{n=i}^{N-|k|-1} x(n)y(n-l)$$

$$r_{xx}(l) = \sum_{n=i}^{N-|k|-1} x(n)x(n-l)$$

$$\boxed{\begin{array}{ll} i = l, k = 0 & \text{if } l \geq 0 \\ i = 0, k = l & \text{if } l < 0 \end{array}}$$

Properties of Crosscorrelation and Autocorrelation Sequences

- **Consider $x(n)$, $y(n)$ and form: $ax(n) + by(n - l)$**
  - a, b arbitrary constants
  - l is some shift
- **Energy of the linearly combined signal:**

$$\sum_{n=-\infty}^{\infty}\left[ax(n)+by(n-l)\right]^2 = a^2\sum_{n=-\infty}^{\infty}x^2(n)+b^2\sum_{n=-\infty}^{\infty}y^2(n-1)+2ab\sum_{n=-\infty}^{\infty}x(n)y(n-l)$$
$$= a^2 r_{xx}(0)+b^2 r_{yy}(0)+2ab r_{xy}(l)$$

- **Note: $r_{xx}(0) = E_x$ is the energy of $x(n)$. Similarly for $r_{yy}(0) = E_y$ is energy of $y(n)$.**
- **If $b \neq 0$, get the quadratic**
- **Its discriminant is non-positive:** $r_{xx}(0)\left(\frac{a}{b}\right)^2 + 2r_{xy}(l)\left(\frac{a}{b}\right) + r_{yy}(0) \geq 0$

$$4\left[r_{xy}^2(l) - r_{xx}(0)r_{yy}(0)\right] \leq 0$$

- **Therefore:**

$$\boxed{\left|r_{xy}(l)\right| \leq \sqrt{r_{xx}(0)r_{yy}(0)} = \sqrt{E_x E_y}}$$

$$\boxed{\left|r_{xx}(l)\right| \leq r_{xx}(0) = E_x}$$

Upper bounds on cross- and autocorrelation sequences

Properties of Crosscorrelation and Autocorrelation Sequences

- **Autocorrelation sequence attains its maximum at zero lag**
- **If any or both of signals are scaled, shape of cross-correlation sequence does not change**
- **Since scaling is unimportant, can normalize $r_{xy}(l)$ to the range -1 to 1:**

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)} \qquad \rho_{xy}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}}$$

$$\left|\rho_{xx}(l)\right| \le 1 \qquad \left|\rho_{xy}(l)\right| \le 1$$

- **Other properties:**

$$r_{xy}(l) = r_{yx}(-l)$$

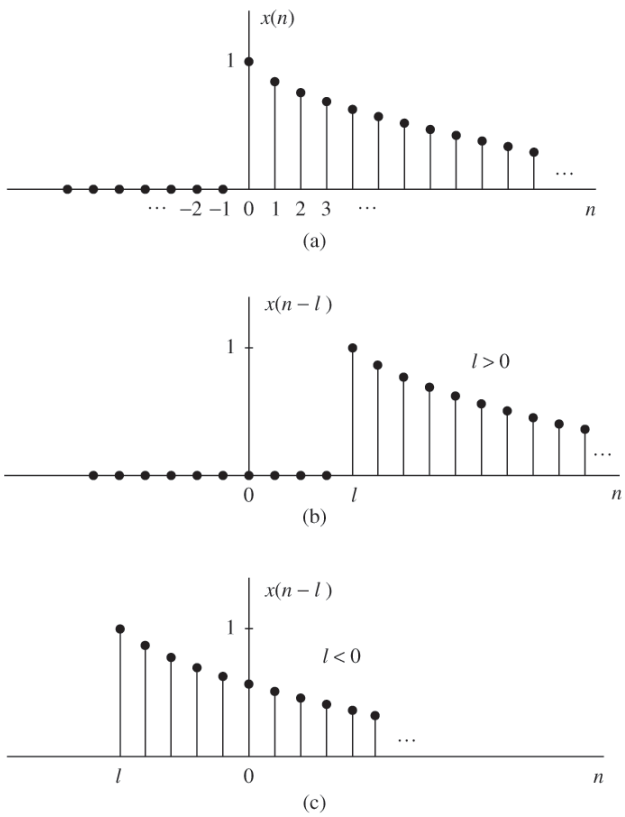
$$r_{xx}(l) = r_{xx}(-l) \Rightarrow \text{hence auto-correlation sequence is even}$$

Example

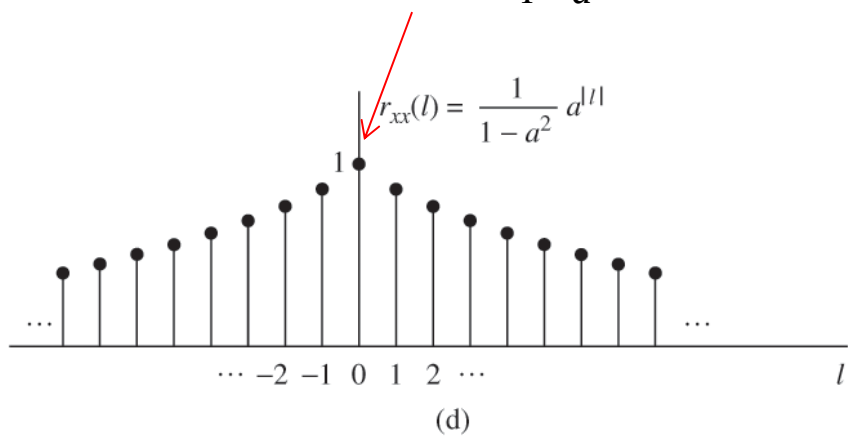
$$x(n) = a^n u(n), \quad 0 < a < 1$$

$$\text{Case } l \geq 0: \quad r_{xx}(l) = \sum_{n=1}^{\infty} x(n)x(n-l) = \sum_{n=1}^{\infty} a^n a^{n-l} = a^{-l} \sum_{n=1}^{\infty} \left(a^2\right)^n \to \frac{1}{1-a^2} a^l$$

$$\text{Case } l < 0: \quad r_{xx}(l) = \sum_{n=0}^{\infty} x(n)x(n-l) = \sum_{n=0}^{\infty} a^n a^{n-l} = a^{-l} \sum_{n=0}^{\infty} \left(a^2\right)^n \to \frac{1}{1-a^2} a^{-l}$$



$$r_{xx}(0) = \frac{1}{1-a^2}$$



$$\rho_{xy}(l) = a^{|l|}$$

Correlation of Periodic Sequences

- **For power signals *x*(*n*), *y*(*n*), the correlation functions are defined as:**

$$r_{xy}(l) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)\,y(n-l), \quad l = 0, \pm 1, \pm 2, \ldots$$

$$r_{xx}(l) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)\,x(n-l), \quad l = 0, \pm 1, \pm 2, \ldots$$

- **In particular, for periodic signals with period N, the above averages are identical to averages over a single period:**

$$r_{xy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\,y(n-l), \quad l = 0, \pm 1, \pm 2, \ldots$$

$$r_{xx}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\,x(n-l), \quad l = 0, \pm 1, \pm 2, \ldots$$

  – Both $r_{xy}(l)$ and $r_{xx}(l)$ are periodic correlation sequences with period N

- **In practical applications, correlation is used to identify periodicities in an observed signal which may be corrupted by random interference**

Correlation of Periodic Sequences (cont'd)

- **Let $y(n) = x(n) + w(n)$ where $x(n)$ is a periodic sequence of unknown period N, and $w(n)$ is additive random interference**
- **Suppose we observe M samples of $y(n)$ for $0 \le n \le M - 1$, where $M >> N$. Also we can assume that $x(n) = 0$ for $n < 0$ and $n \ge M$**

$$
\begin{aligned}
r_{xx}(l) &= \frac{1}{M}\sum_{n=0}^{M-1} y(n)\,y(n-l) \\
&= \frac{1}{M}\sum_{n=0}^{M-1}\left[x(n)+w(n)\right]\left[x(n-l)+w(n-l)\right] \\
&= r_{xx}(l) + r_{xw}(l) + r_{wx}(l) + r_{ww}(l)
\end{aligned}
$$

- **Since $x(n)$ is periodic, $r_{xx}(l)$ will show peaks at $l = 0, 2N, 4N, \ldots$ As l approaches M, the peaks are reduced in amplitude since M is finite and many of the terms $x(n)x(n - l)$ are zero**
  - Consequently should keep $l \le M/2$
- **The other terms $r_{xw}(l)$, $r_{wx}(l)$, $r_{ww}(l)$ are relatively small**

Example

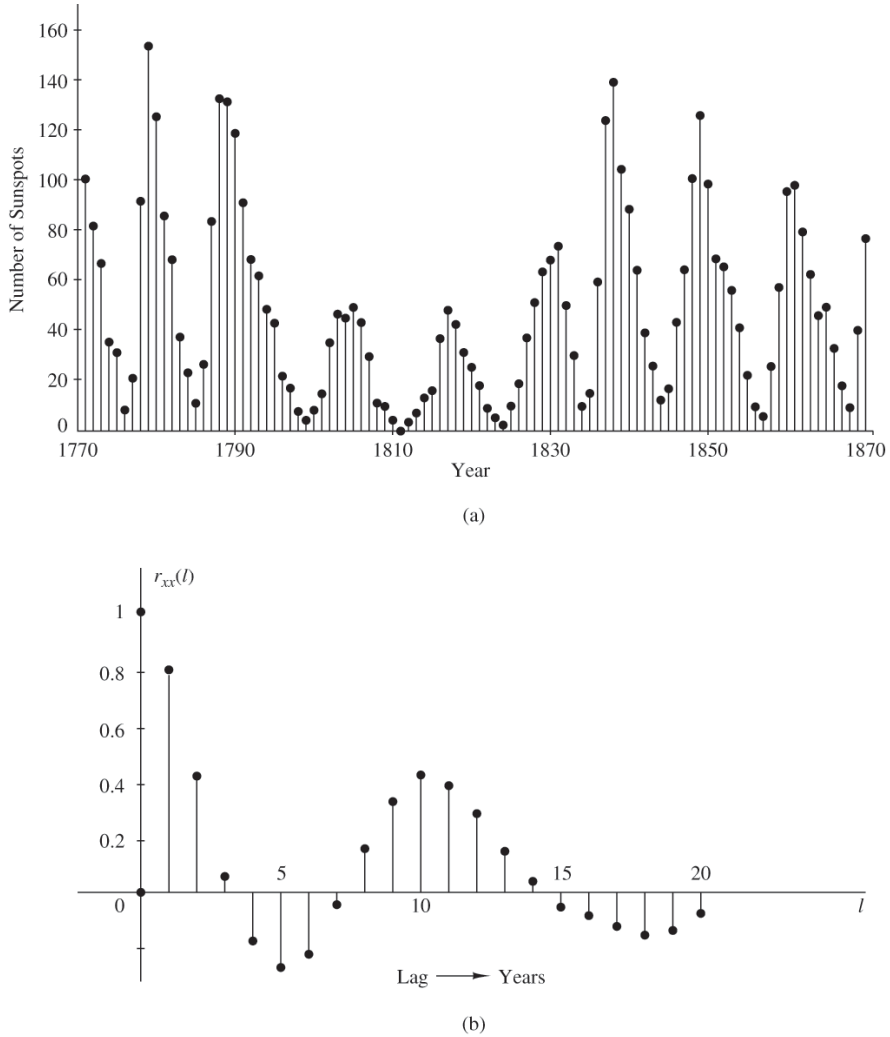


Figure 2.6.3 Identification of periodicity in the Wölfer sunspot numbers: (a) annual Wölfer sunspot numbers; (b) normalized autocorrelation sequence.

Input-Output Correlation Sequences

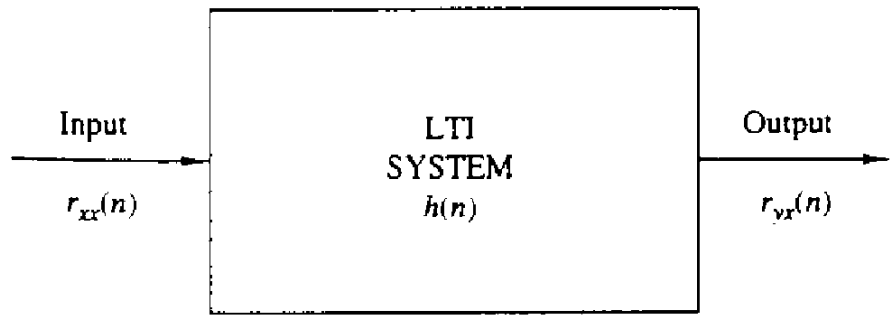
- **Let $x(n)$ be a signal with know $r_{xx}(l)$, applied to an LTI system with impulse response $h(n)$:**

$$y(n) = h(n) * x(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

- **The cross-correlation between the input and output signal is**

$$\begin{aligned} r_{yx}(l) &= y(l) * x(-l) \\ &= h(l) * \left[x(l) * x(-l)\right] \\ &= h(l) * r_{xx}(l) \end{aligned}$$

$$r_{xy}(l) = h(-l) * r_{xx}(l)$$



- **Autocorrelation of output:**

$$\begin{aligned} r_{yy}(l) &= y(l) * y(-l) \\ &= \left[h(l) * x(l)\right] * \left[h(-l) * x(-l)\right] \\ &= \left[h(l) * h(-l)\right] * \left[x(l) * x(-l)\right] \\ &= r_{hh}(l) * r_{xx}(l) \end{aligned}$$

For $l = 0$ $\longrightarrow$

$$r_{yy}(0) = \sum_{k=-\infty}^{\infty} r_{hh}(k)\,r_{xx}(k)$$

Total energy or power of the output signal