EECE 491: Discrete-time Signal Processing

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Lecture 2: Review of Signals, Systems, and Signal Processing

Announcements

- **Reading**
  - Chapter 1, Proakis

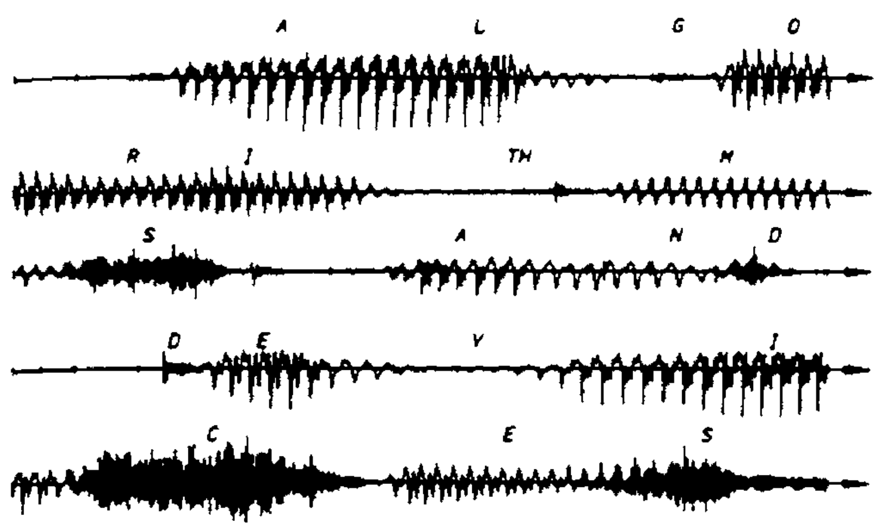
Outline

- **Signals and signal types**
  - Continuous-time vs. discrete-time frequency
  - Harmonically-related complex exponentials
- **SP systems**
- **Sampling**
- **A/D and D/A**
- **Quantization**
  - SQNR

Signals

- **Signal: Any physical quantity that varies with time, space or other variables**
  - $s_1(t) = 5t$
  - $s_2(t) = 20t^2$
  - $s(x,y) = 3x + 2xy + 10y^2$
- **Other examples include signals that cannot be expressed by a formula**
  - Ex: speech signal
  - Approximate as a sum of sinusoids

$$\sum_{i=1}^{N} A_i(t)\sin\left(2\pi F_i(t)t + \theta_i(t)\right)$$



- **Other natural signals:**
  - ECG signal, EEG signal

Signal Processing Systems

- **Any system that performs operations on a signal**
  - Ex: A filter that reduces noise and interference corrupting an info bearing signal
  - As the signal passes through the system, it gets processed

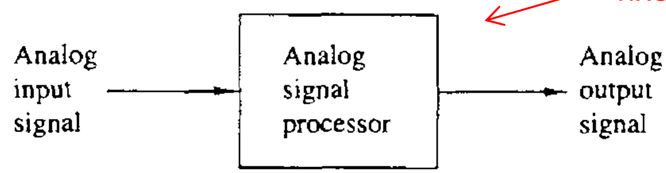
- **Signal processing**
  - Analog signal processing
  - Optical signal processing
  - Digital signal processing: Includes both hardware (logic circuits) and software (a program that runs on a DSP) realizations

- **Focus is on DSP**
  - Operations performed are specified mathematically
  - The method of implementing a system that performs these operations is called a DSP algorithm
  - Many ways to implement the algorithm
  - Interested in devising *computationally efficient*, *fast*, and *easy to implement* algorithms

Basic Elements of a DSP System

- **Most signals in science and engineering are analog in nature**
  - Functions of continuous variables in time or space
  - Take values in a continuous range
- **Can process directly using by an analog system**
  - Ex: Filter, frequency analyzer, frequency multiplier

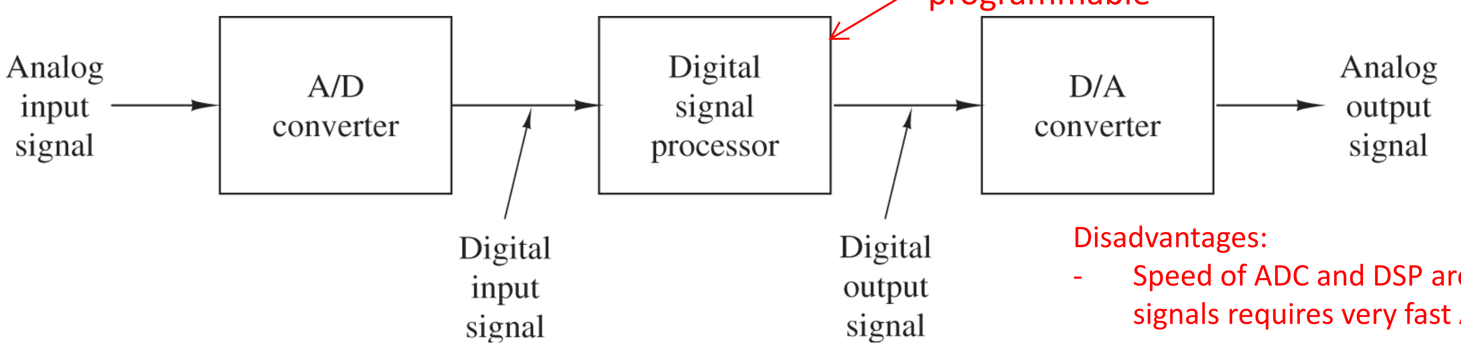
fixed, hard-wired



- **Alternatively, can process using a DSP**
  - Need analog-to-digital converter (A/D or ADC)
  - Digital-to-analog converter (D/A or DAC)

Advantages:
- Better accuracy
- Stored without loss of fidelity
- Cheaper

programmable



Disadvantages:
- Speed of ADC and DSP are limited. Wideband signals requires very fast ADCs and DSPs

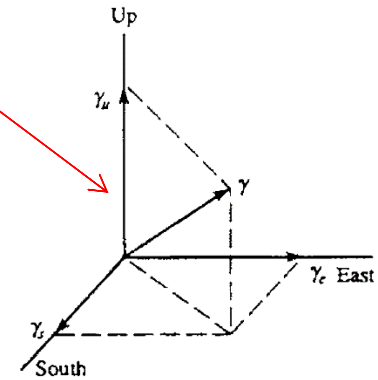
Classification of Signals

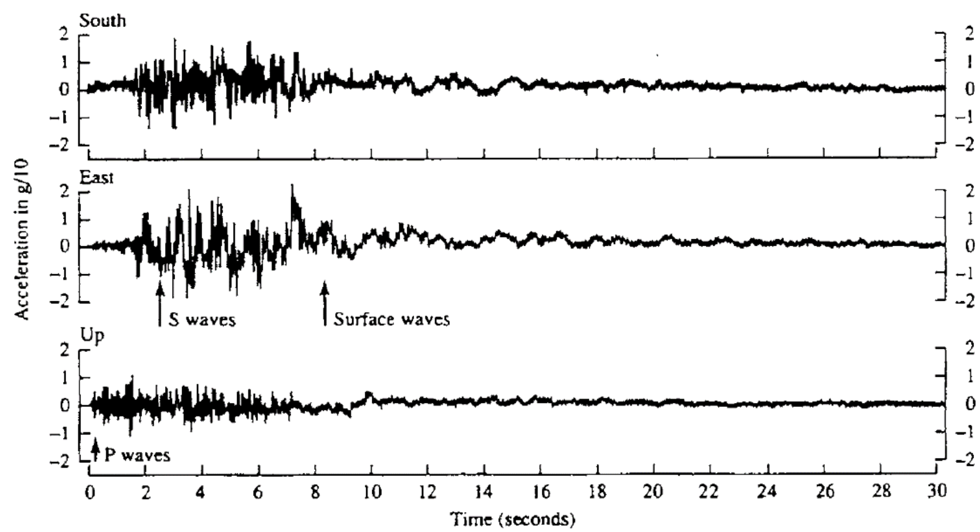
- **Multichannel** and **multidimensional** signals
- **Continuous-time** versus **discrete-time** signals
- **Continuous-valued** versus **discrete-valued** signals
- **Deterministic** versus **random** signals

Multichannel and Multidimensional Signals

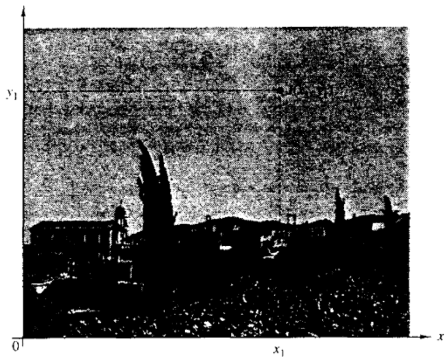
- 1D real-valued signal: $s_1(t) = A\ sin(3\pi t)$
- 1D complex-valued signal: $s_2(t) = Ae^{j3\pi t} = A\ cos(3\pi t) + j\ A\ sin(3\pi t)$
- Seismic wave signal: multi-dimensional vector signal

$$\mathbf{S}(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{bmatrix}$$





- Ex: 2D picture signal
  - $I(x,y)$ intensity function

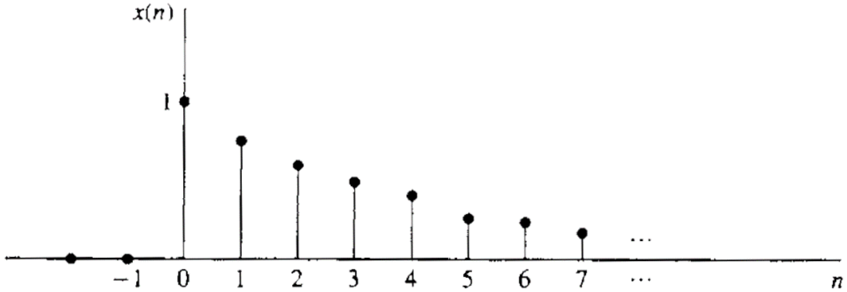


- Ex: Video signal: red, green, blue intensities varying in time

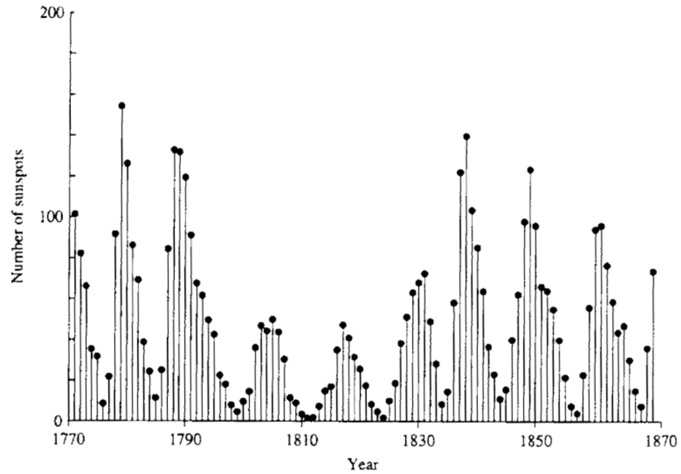
$$\mathbf{I}(x,y,t) = \begin{bmatrix} I_r(x,y,t) \\ I_g(x,y,t) \\ I_b(x,y,t) \end{bmatrix}$$

Continuous-Time (CT) vs. Discrete-Time (DT) Signals

- **Example**: $x_a(t) = 0.8^t$ **if** $t > 0$ **and 0 otherwise**
- $x[n] \triangleq x_a(nT) = 0.8^{nT}$



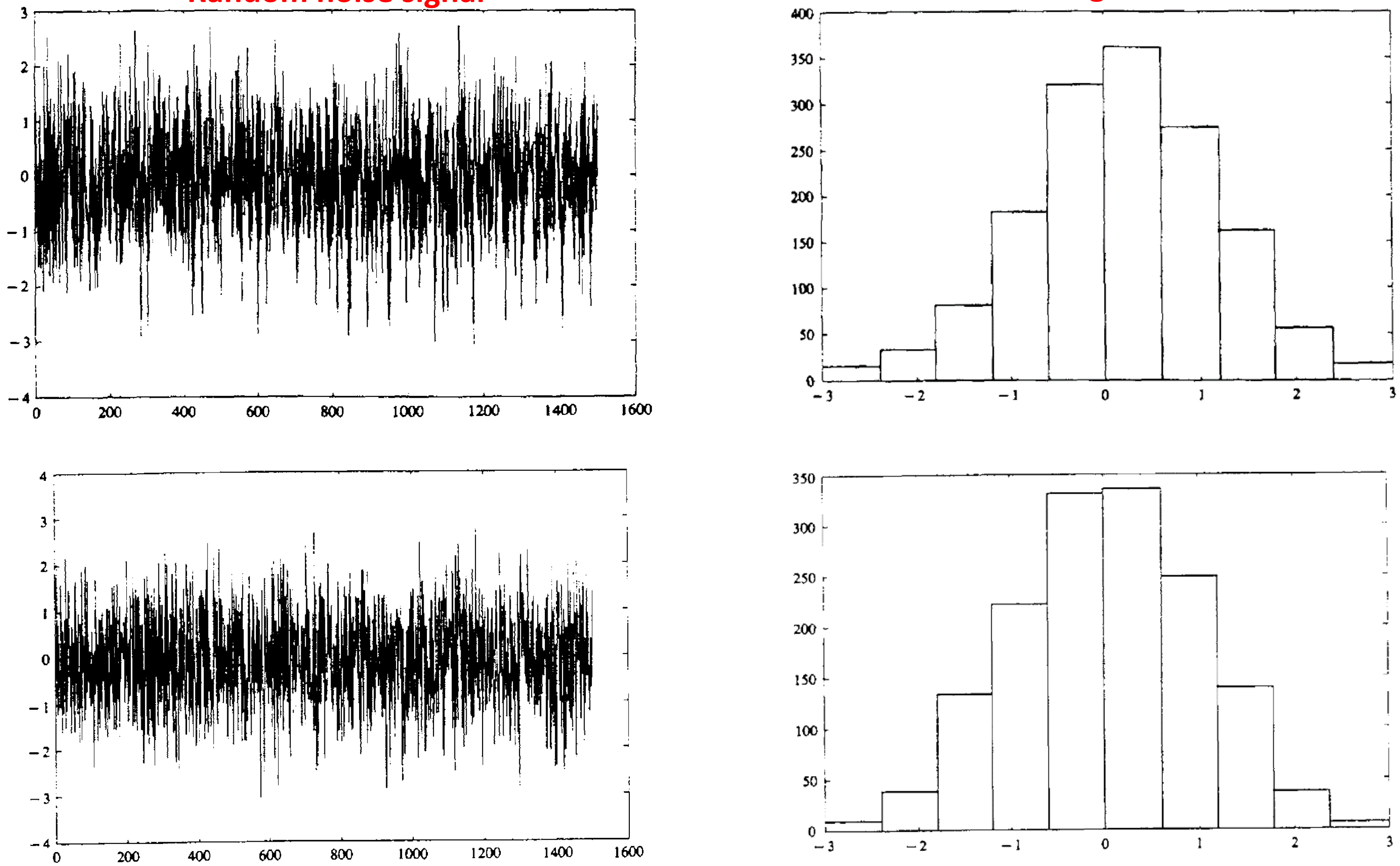
- **The discrete-time signal was obtained by sampling a continuous-time signal at equally spaced time instances** $t_n = nT$
- **DT signals can arise naturally by accumulating a variable over a period of time**



Deterministic vs Random signals

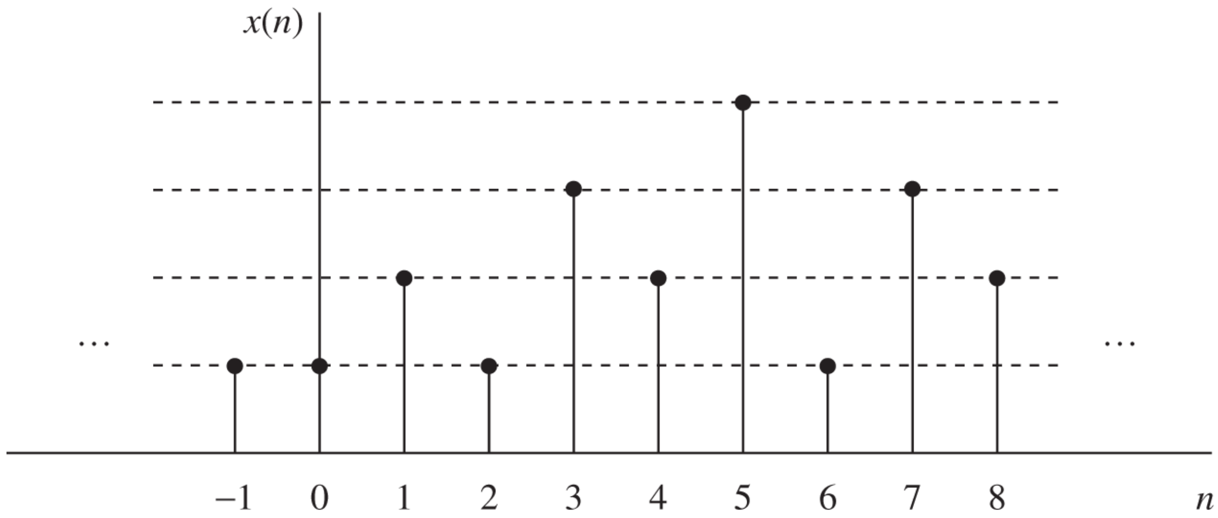
- **Signals sometimes cannot by fully specified because they evolve in an unpredictable way**
  - Analyzed and described using statistical techniques

Random noise signal **Histogram**



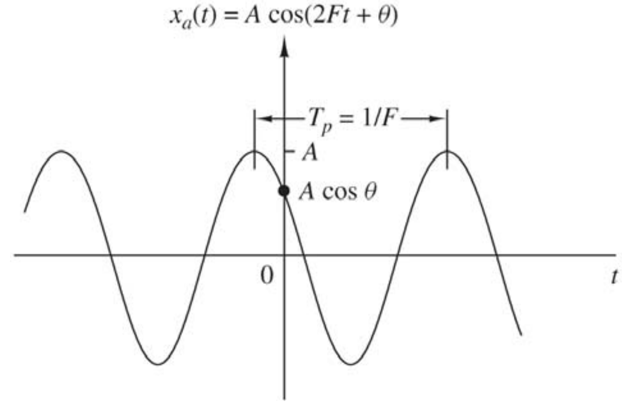
Continuous-Valued vs Discrete-Valued Signals

- **Values of a CT and DT signals can continuous or discrete**
- **Discrete valued DT signals are called digital signals**
- **Digital signals are obtained by quantizing a sampled CT signal**

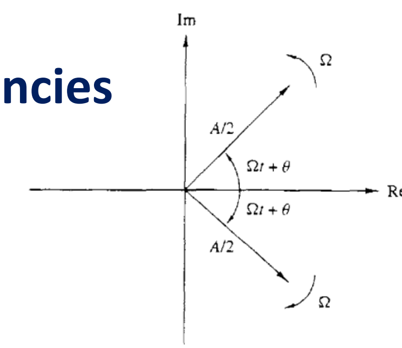


Frequency of a CT Signal

- $x_a(t) = A\cos(\Omega t+\theta), \quad -\infty < t < \infty$
  - A: Amplitude
  - Ω: Frequency in rad/sec; $\Omega=2\pi F$, F is frequency in cycles/sec (Hz)
  - θ: Phase in radians

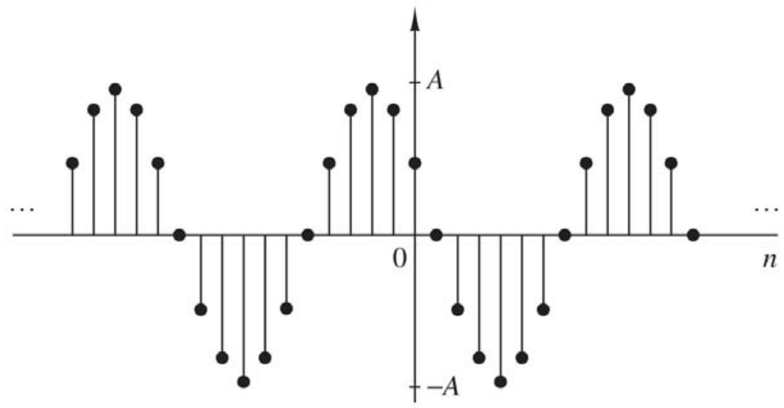


- **Properties of analog sinusoidal signals**
  - For any fixed F, $x_a(t)$ is periodic with $T_p = 1/F$ being its fundamental period
  - CT sinusoidal signals with different frequencies are themselves different
  - Increasing F results in an increase in the rate of oscillation
- **These properties apply to class of complex-exponential signals as well**
  - $x_a(t) = Ae^{j(\Omega t+\theta)} = A\cos(\Omega t+\theta) + jA\sin(\Omega t+\theta)$
- **For mathematical convenience we use negative frequencies**
  - $\cos(\Omega t+\theta) = A/2\ e^{j(\Omega t+\theta)} + A/2\ e^{j(\Omega t-\theta)}$
  - Phasor representation



Frequency of a DT Signal

- $x[n] = x_a(nT) = \cos(\omega n + \theta), \quad -\infty < n < +\infty$
  - A: Amplitude
  - $\omega = 2\pi f$: Frequency in rad/sample; f is frequency in cycles/sample (Hz)
  - θ: Phase in radians



1. **Properties of DT sinusoidal signals: A DT sinusoid is periodic with period *N* if *f* is rational.**
   - Periodic: Smallest *N* that satisfies *x*[*n*+*N*] = *x*[*n*] for all *n* is called fundamental period
   - Ex: for a sinusoid with freq f_0: $\cos(2\pi f_0(N+n)+\theta) = \cos(2\pi f_0 n + \theta) \quad \Rightarrow \exists\, k \in \mathbb{Z} : 2\pi f_0 N = 2k\pi$
   - Ex: $f_1 = 31/60 \Rightarrow N_1 = 60$
   - Ex: $f_2 = 30/60 \Rightarrow N_2 = 2$

Properties of DT Sinusoidal Signals (cont'd)

2. DT sinusoids whose frequencies are integer multiples of 2π are identical

$$\cos\left(\left(\omega_0 + 2\pi\right)n + \theta\right) = \cos\left(\omega_0 n + \theta\right)$$

- **Hence, all sinusoidal sequences $x_k\ (n)= A\cos(\omega_k n+\theta)$, k = 0,1,… are identical when $\omega_k = \omega_0 + 2\pi k$, where $-\pi \le \omega_0 \le \pi$**

- **Any sequence resulting from a sinusoid with freq $|\omega| > \pi$ is identical to a sequence obtained from a sinusoidal signal with $|\omega| < \pi$**

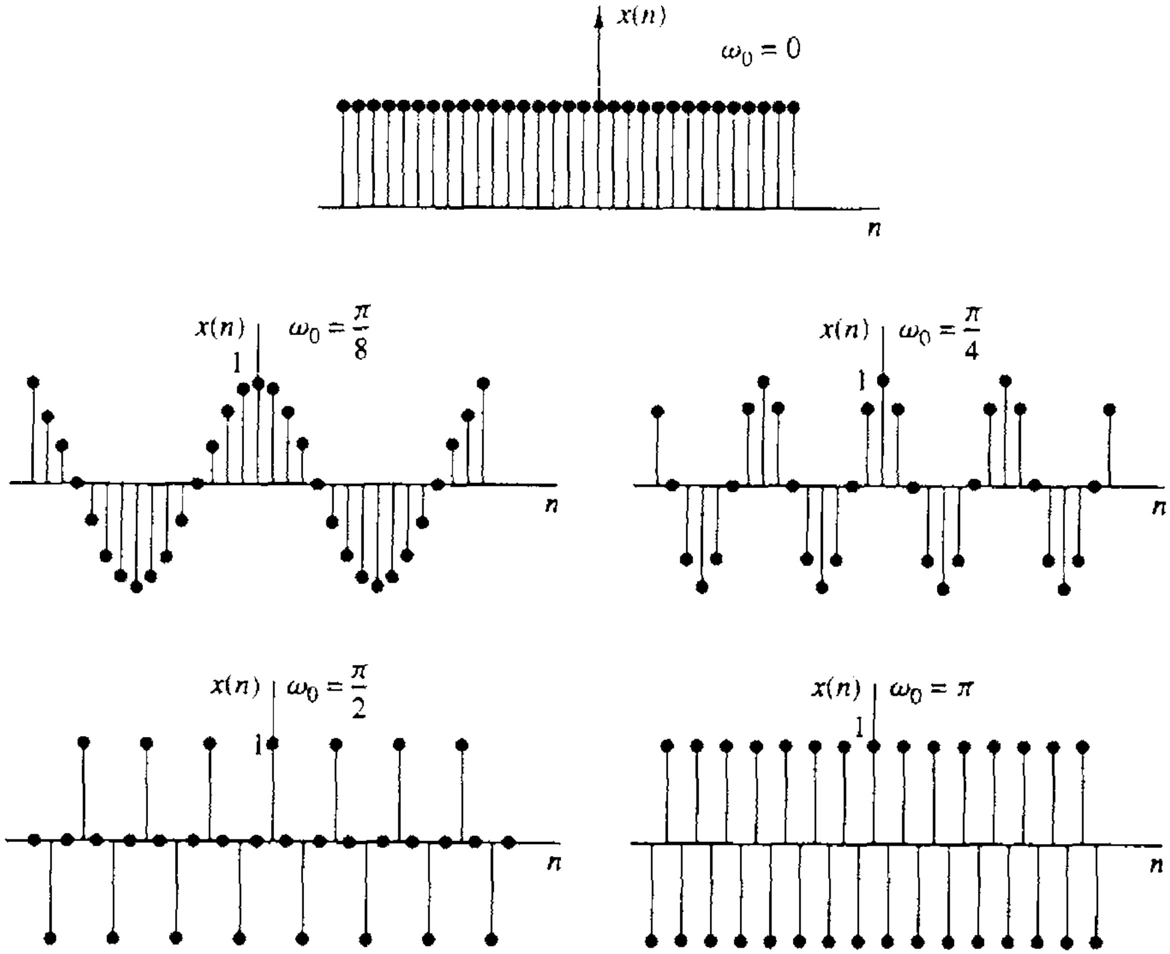
- **Sinusoids with $|\omega| > \pi$ are called aliases**

- **Frequencies in the range $-\pi \le \omega \le \pi$ or $-1/2 \le f \le 1/2$ are unique**
  - Frequencies in the range $|\omega| > \pi$ or $|f| > 1/2$ are aliases

- **Ex: for $\pi \le \omega_0 \le 2\pi$, consider two sinusoids with freqs $\omega_1 = \omega_0$ and $\omega_2 = 2\pi - \omega_0$**
  - $x_1[n] = A\cos(\omega_1 n + \theta)$
  - $x_2[n] = A\cos(\omega_2 n+\theta) = A\cos((2\pi-\omega_0)n+\theta) = x_1[n]$

Properties of DT Sinusoidal Signals (cont'd)

3. The highest rate of oscillation is attained when $\omega = \pm\pi$ or $f = \pm 1/2$



Harmonically-Related Complex Exponentials (HRCEs)

- **Harmonically-Related Complex Exponentials: Set of periodic complex exponentials with fundamental frequencies multiple of a single frequency**

- **CT exponentials:** $s_k(t) = e^{jk\Omega_0 t} = e^{j2\pi k F_0 t}, \quad k = 0, \pm 1, \pm 2, \cdots$
  - Fundamental period: $1/(kF_0) = T_p/k$
  - Fundamental freq: kF_0
  - Periodic signals with period T_p/k and hence also with period T_p.
  - F_0 can take any value and all members of the set are distinct

- **Basic signals are a linear combination of HRCEs:**

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k s_k(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\Omega_0 t}$$

  - c_k: arbitrary complex constants
  - $x_a(t)$ is periodic with fundamental period $T_p = 1/F_0$.
  - Representation is called Fourier series expansion of $x_a(t)$ and c_k's are Fourier series coefficients.
  - $s_k(t)$ is called the k^{th} harmonic

Discrete-Time Exponentials

- **A DT complex exponential is periodic if its relative frequency is a rational number, choose $f_0 = 1/N$**
- **Define the set of DT-HRCEs as** $s_k[n] = e^{j2\pi k f_0 n}, k = 0, \pm 1, \pm 2, \cdots$
- **Note that** $s_{k+N}[n] = e^{j2\pi k f_0 (n+N)} = e^{j2\pi n} s_k[n] = s_k[n]$
- **Hence there are only *N* distinct DT-HRCEs in the above set**
- **All members have a common period of *N* samples**
- **Choose any consecutive *N* complex exponentials to form the set of HRCEs**
  - Most convenient choice: $s_k[n] = e^{j2\pi kn/N}, \quad k = 0, 1, \cdots, N-1$
- **Linear combination:**

$$x[n] = \sum_{k=0}^{N-1} c_k s_k[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

  - Periodic with period *N*
  - Called Fourier series expansion for a periodic DT signal, with Fourier coefficients $\{c_k\}$
  - $s_k[n]$ is called the k^{th} harmonic

Example

- **Assume you have stored in memory of a DSP one cycle of the sinusoid:**
  - $x[n] = \sin(2\pi n/N+\theta)$ where $\theta = 2\pi q/N$, where q and N are integers
  - i.e., the samples $x[0]$, $x[1]$, …, $x[N\text{-}1]$ are stored in the processor's memory
- **Q1: Determine how this table of values can be used to obtain values of harmonically related sinusoids having the same phase**
  - That is find $x_2[n] = \sin(2\pi n2/N+\theta)$, $x_3[n] = \sin(2\pi n3/N+\theta)$, …, $x_k[n] = \sin(2\pi nk/N+\theta)$
- **Answer: Let $x_k[n] = \sin(2\pi nk/N+\theta)$ having frequency $f_k = k/N$ which is harmonically related to x[n].**
  - $x_k[n] = \sin(2\pi nk/N+\theta) = x[kn \bmod N]$
  - Read off values from memory for every n to determine all $x_k[n]$
  - *Ex: Assume N = 10, and k* = 2. The 10 values of the harmonic sequence x_2[n] are:

$x_2[0] = x[0]$	$x_2[1] = x[2]$	$x_2[2] = x[4]$	$x_2[3] = x[6]$	$x_2[4] = x[8]$	(two periods)
$x_2[5] = x[0]$	$x_2[6] = x[2]$	$x_2[7] = x[4]$	$x_2[8] = x[6]$	$x_2[9] = x[8]$	

  - *Ex: Assume N = 10, and k* = 3. The 10 values of the harmonic sequence $x_3[n]$ are:

$x_3[0] = x[0]$	$x_3[1] = x[3]$	$x_3[2] = x[6]$	$x_3[3] = x[9]$	
$x_3[4] = x[2]$	$x_3[5] = x[5]$	$x_3[6] = x[8]$	$x_3[7] = x[1]$	(one period)
$x_3[8] = x[4]$	$x_3[9] = x[7]$			

Example (cont'd)

- **Q2: Determine how this table of values can be used to obtain sinusoids having the same frequency but different phase.**
  - That is given $x_k[n]=\sin(2\pi nk/N+\theta), \quad \theta=2\pi q/N$
  - Determine $y_k[n]=\sin(2\pi nk/N+\theta'), \quad \theta'=\theta+2\pi/N$

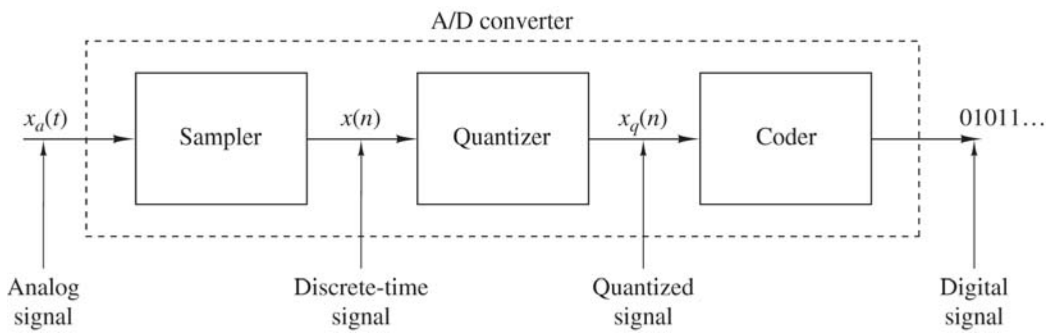
- **Answer:** $$\begin{aligned} y_k[n] &= \sin(2\pi nk/N+\theta+2\pi k/N) \\ &= \sin(2\pi nk/N+\theta+2\pi k/N) \\ &= \sin(2\pi(n+1)k/N+\theta) \\ &= x_k[n+1] \end{aligned}$$

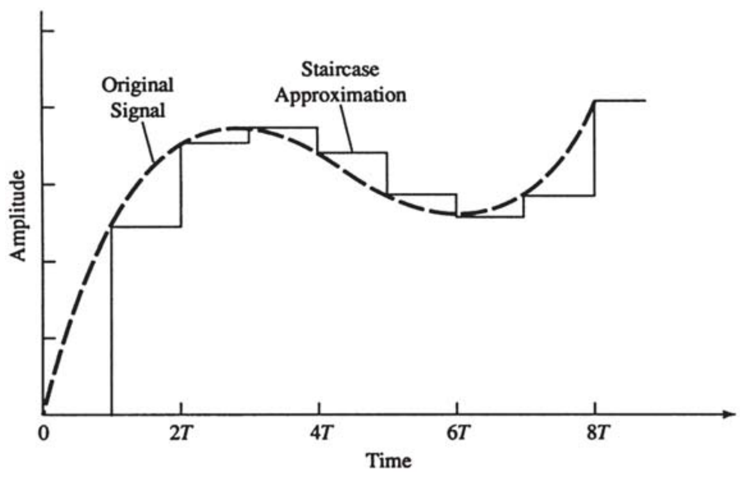
  - The same applies for any other phase $\theta'=\theta+2\pi m/N$

A/D and D/A Conversion

A/D and D/A Conversion

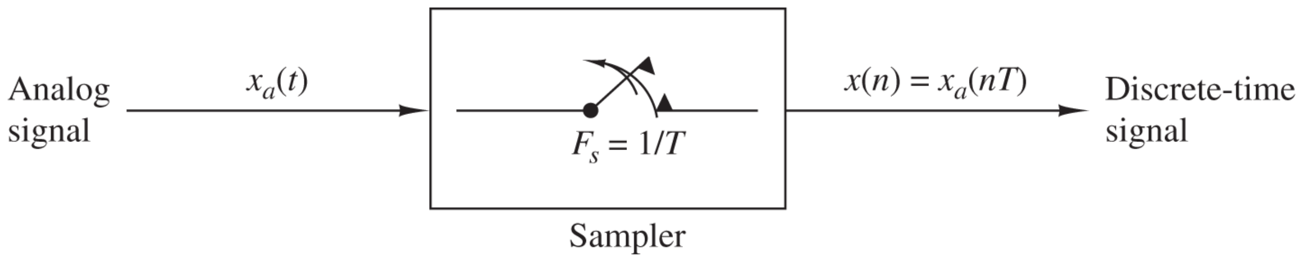
- **Three steps:**
  - **Sampling**: Take samples of a CT signal at discrete time instances: $x[n]$=$x_a(nT)$, T is the sampling period, and f = $1/T$ is the sampling rate.
    - Uniform or periodic sampling
  - **Quantization**: Convert a DT continuous-valued signal into a digital signal.
    - $x_q[n]$ = $Q(x[n])$, where $|x[n]−x_q[n]|$ is the quantization error
  - **Coding**: Each discrete value $x_q[n]$ is mapped to a b-bit binary number

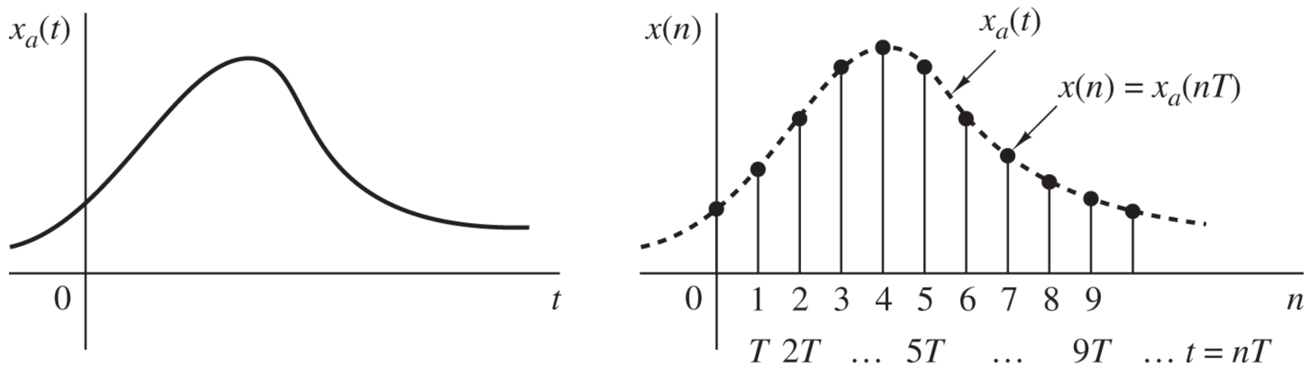




Sampling of Analog Signals

- $x[n] = x_a(nT), \; -\infty < n < \infty$
- **Sampling rate: $F_s = 1/T$ (samples per sec, Hertz)**
- **Relationship between *t* and *n*: $t = nT = n/F_s$**
- **Hence there exists a relationship between frequency *F* (analog signal) and *f* (DT signal)**





Sampling of Analog Signals (cont'd)

- $x_a(t) = A\cos(2\pi Ft+\theta)$, **when sampled at** $F_s = 1/T$ **gives**

$$x_a(nT) = x[n] = A\cos(2\pi nF/F_s + \theta) \rightarrow A\cos(2\pi nf + \theta)$$

$$\Rightarrow f = \frac{F}{F_s}$$

$$\Rightarrow \omega = 2\pi f = 2\pi\frac{F}{F_s} = 2\pi F \times \frac{1}{F_s} = \Omega T$$

- **CT:** $-\infty < F < \infty \qquad -\infty < \Omega < \infty$
- **DT:** $-1/2 < f < 1/2 \qquad -\pi < \omega < \pi$

Continuous-time signals		Discrete-time signals
$\Omega = 2\pi F$ $\frac{\text{radians}}{\text{sec}}$ Hz	$\omega = \Omega T,\ f = F/F_s$ →	$\omega = 2\pi f$ $\frac{\text{radians}}{\text{sample}}$ $\frac{\text{cycles}}{\text{sample}}$
	← $\Omega = \omega/T,\ F = f \cdot F_s$	$-\pi \le \omega \le \pi$ $-\frac{1}{2} \le f \le \frac{1}{2}$
$-\infty < \Omega < \infty$ $-\infty < F < \infty$		$-\pi/T \le \Omega \le \pi/T$ $-F_s/2 \le F \le F_s/2$

- **Therefore:**

$$-\frac{1}{2} \le \frac{F}{F_s} \le \frac{1}{2} \quad \Rightarrow \quad -\frac{1}{2T} = -\frac{F_s}{2} \le F \le \frac{F_s}{2} = \frac{1}{2T}$$

- **Hence: The frequency of a CT sinusoid when sampled at a rate $F_s = 1/T$ must fall in the range $-Fs/2 \le F \le Fs/2$**
- **Map an infinite frequency range of a CT signal into a finite frequency range for the DT signal**
- $F_{max} = F_s/2$

$$x[n] = x_a(nT) = \cos(\omega n + \theta), \quad -\infty < n < +\infty$$

Example

- **Consider two CT signals sampled at F_s = 40Hz**
  - $x_1(t) = cos2\pi(10)t$
  - $x_2(t) = cos2\pi(50)t$

- **Corresponding DT signals are:**
  - $x_1(n) = cos2\pi(10/40)n = \cos(\pi/2\ n)$
  - $x_2(n) = cos2\pi(50/40)n = \cos(5\pi/2\ n) = \cos(\pi/2\ n)$
  - Therefore $x_1(n) = x_2(n)$

- **The two DT signals are identical and hence they are indistinguishable**

- **We say that the frequency 50Hz is an alias of the frequency F_1 = 10Hz at the sampling rate of 40 samples/sec.**

- **Also all frequencies $F_1 + 40k$, k = 0, 1, 2, … are aliases as well**

Sampling of Analog Signals (cont'd)

$$x_a(t) = A\cos(2\pi F_0 t + \theta)$$

sample @ $F_s = 1/T$

$$x_a[n] = A\cos(2\pi f_0 n + \theta), \quad f_0 = \frac{F_0}{F_s}$$

Two cases:

- **Case I: If $-\frac{F_s}{2} \le F_0 \le \frac{F_s}{2}$, then f_0 is in the range $-\frac{1}{2} \le f_0 \le \frac{1}{2}$. The relationship between F_0 and f_0 is one to one, and it is possible to reconstruct $x_a(t)$ from $x[n]$.**
- **Case II: Otherwise, for sinusoids $x_a(t) = A\cos(2\pi F_k t + \theta)$ with frequencies $F_k = F_0 + kF_s$, $k = \pm 1, \pm 2, \ldots$ If they are sampled at F_s, it is clear that F_k is outside the fundamental range. Hence the sampled signal in this case is**

$$x_a(t) = A\cos(2\pi F_k t + \theta)$$

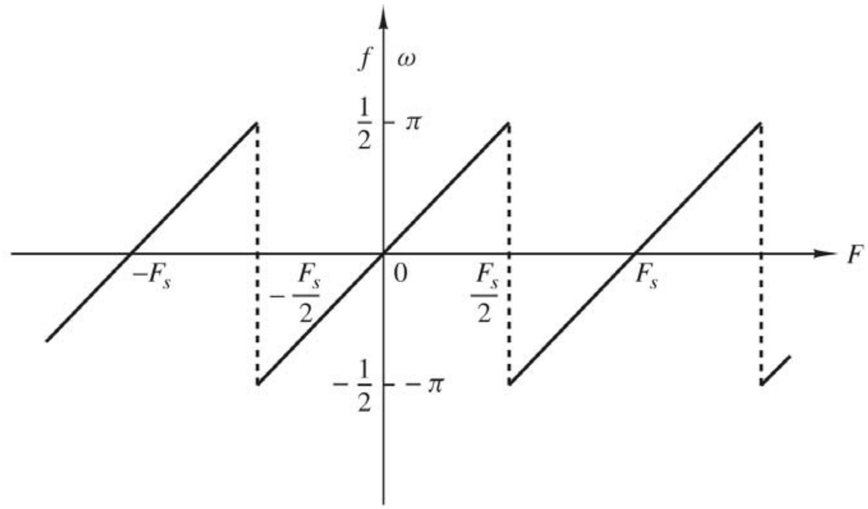
sample @ $F_s = 1/T$

$$x[n] = A\cos\left(2\pi \frac{F_0 + kF_s}{F_s} n + \theta\right)$$

$$= A\cos\left(2\pi \frac{F_0}{F_s} n + \theta\right)$$

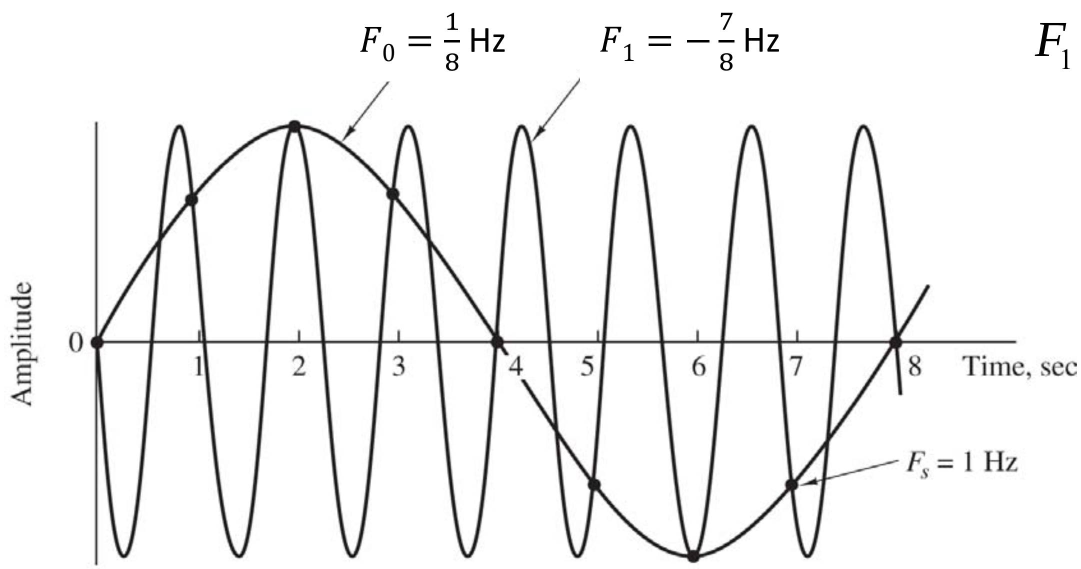
$$= A\cos(2\pi f_0 n + \theta)$$

Sampling of Analog Signals (cont'd)



$$F_k = F_0 \pm kF_s$$

$$F_1 = F_0 - 1 = -\frac{7}{8}$$



Example

- Consider the analog signal $x_a(t) = 3cos(100\pi t)$
- Q1: Determine the minimum sampling rate required to avoid aliasing.
- A1: $F_{s,min} = 2 \times F = 2 \times 50 = 100 Hz$

- Q2: Suppose $x_a(t)$ is sampled at $F_s = 200$Hz. What is the resulting DT signal?
- A2: $x[n] = 3\cos\left(\frac{100\pi n}{200}\right) = 3\cos\left(\frac{\pi n}{2}\right)$

- Q3: Suppose $x_a(t)$ is sampled at $F_s = 75$Hz. What is the resulting DT signal?
- A3: $x[n] = 3\cos\left(\frac{100\pi n}{75}\right) = 3\cos\left(\frac{4\pi n}{3}\right) = 3\cos\left(\frac{2\pi n}{3}\right)$

- Q4: What is the frequency $0 < F < F_s/2$ of an analog sinusoid that yields samples identical to those in Q3?
- A4: For the sampling rate $F_s = 75$ Hz, we have $F = f \times Fs = 75f$. The frequency of the DT signal in Q3 is $f = 1/3$. So the frequency of the CT signal is $F = 25$ Hz and the corresponding sinusoid is $y_a(t) = 3cos(50\pi t)$

The Sampling Theorem

- **Need to know the maximum frequency content of a signal to sample it properly**
- ***Example*:** $x_a(t)=\sum_{i=1}^{N} A_i \sin\left(2\pi F_i(t)t+\theta_i(t)\right)$
  - Speech signal: *N* sinusoids, where F_{max} = 3000 Hz
  - TV signal: *N* sinusoids, where F_{max} = 5 MHz
- **If signal is not band limited to F_{max}, we need to low-pass filter it before sampling**
- **To avoid ambiguities resulting from aliasing, must select**

$$F_s > 2F_{\max}$$

- **In this case, frequency F_i in the above sinusoid maps to**

$$F_i \rightarrow f_i = \frac{F_i}{F_s} \text{ such that } -\frac{1}{2} < f_i < \frac{1}{2}$$

The Sampling Theorem (cont'd)

- **<u>Sampling Theorem</u>: If the highest frequency in an analog signal $x_a(t)$ is $F_{\max}=B$ and the signal is sampled at $F_s > 2F_{\max} \equiv 2B$, then $x_a(t)$ can be exactly recovered from its samples $x_a[n]$ using the interpolation function $g(t)$, and $x_a(t)$ can be written as follows:**

Interpolation function

$$g(t) = \frac{\sin 2\pi Bt}{2\pi Bt}$$

Reconstruction formula

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a\left(\frac{n}{F_s}\right) g\left(t - \frac{n}{F_s}\right)$$

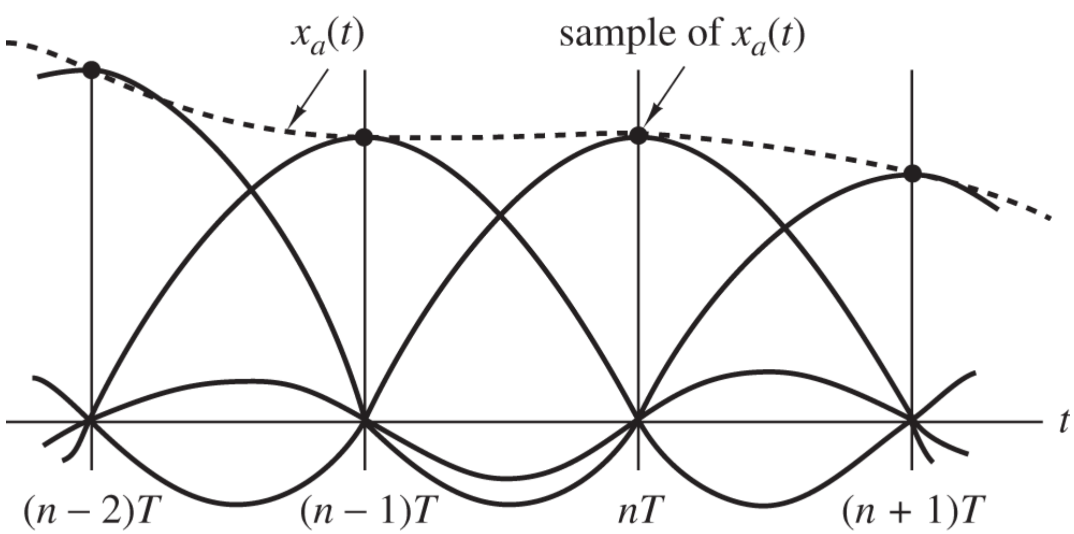
The Sampling Theorem (cont'd)

- **When the sampling is performed exactly at the minimum rate $F_s = 2B$, then**

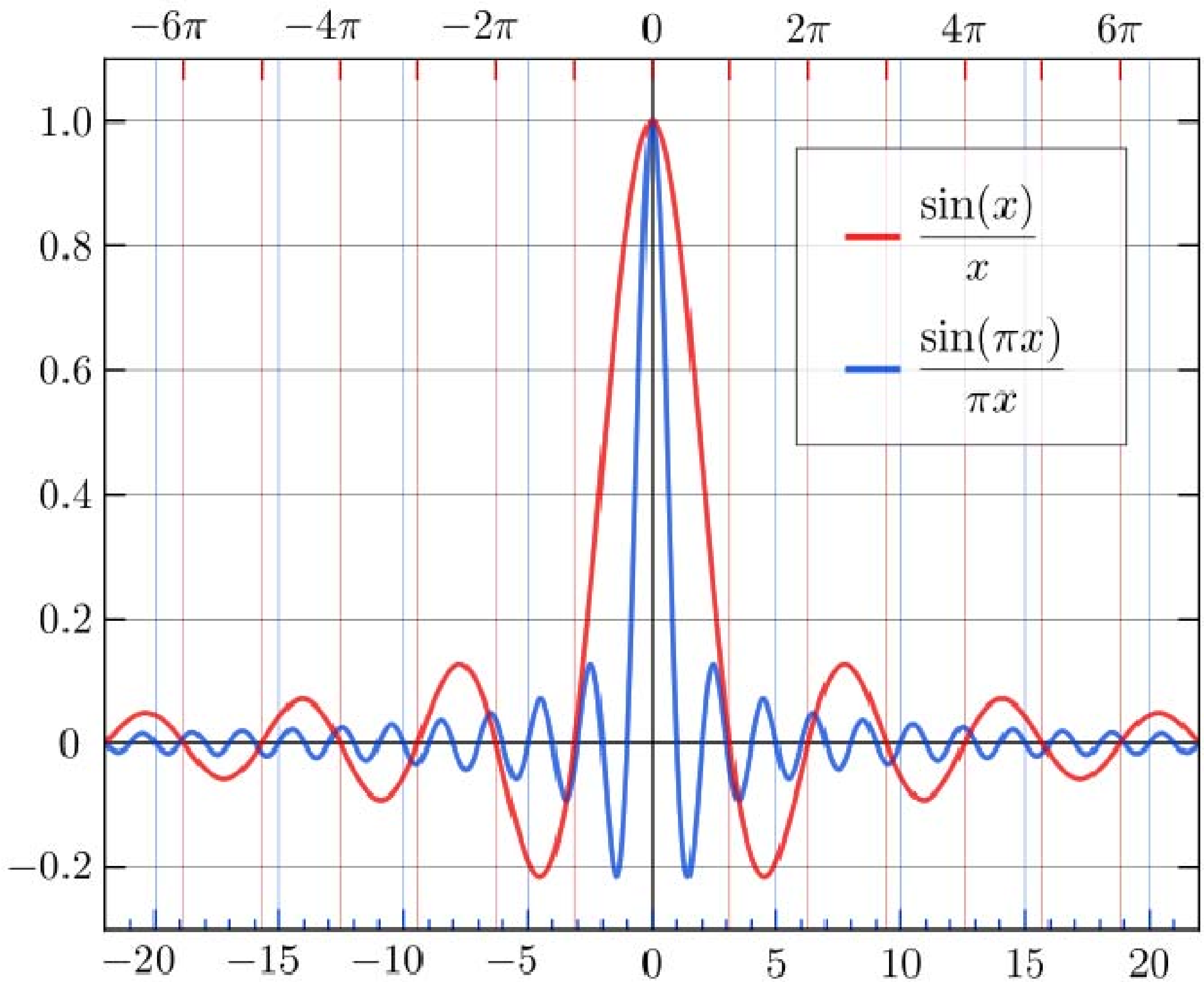
$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a\left(\frac{n}{2B}\right) \frac{\sin 2\pi B(t - n/2B)}{2\pi B(t - n/2B)}$$

Nyquist rate: $F_{Nyquist} = 2F_{\max} = 2B$

Process called Ideal D/A conversion



Sinc Function



zero crossings of normalized sinc occur at non-zero integers

Example

$$x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$$

- Q1: What is the Nyquist rate?
- A1: The frequencies of $x_a(t)$ are F_1 = 1 kHz, F_2 = 3 kHz, F_3 = 6 kHz. Hence F_s must be at least $2F_{\max}$ = 12 kHz which is the Nyquist rate.

- Q2: Assume signal is sampled at F_s = 5000 samp/s. What is the DT signal obtained?
- A2: Note here the choice of F_s is less than the Nyquist rate. Hence aliasing will occur. The maximum frequency that can be represented without aliasing is $F_s/2$ = 2.5 kHz. This can be shown as well as follows:

$$\begin{aligned} x[n] &= x_a(nT) = x_a\left(\frac{n}{F_s}\right) \\ &= 3\cos 2\pi\frac{n}{5} + 5\sin 2\pi\frac{3n}{5} + 10\cos 2\pi\frac{6n}{5} \\ &= 3\cos 2\pi\frac{n}{5} + 5\sin 2\pi\frac{3n}{5} + 10\cos 2\pi\frac{n}{5} \\ &= 13\cos 2\pi\frac{n}{5} - 5\sin 2\pi\frac{2n}{5} \end{aligned}$$

$$\Rightarrow f_1 = \frac{1}{5}, \quad f_2 = -\frac{2}{5}$$

- Q3: What is the analog signal $y_a(t)$ that we can reconstruct from the samples if we use ideal interpolation?
- A3: The freq components present in the signal are $F_1 = f_1 \cdot F_s$ = 1 kHz and $F_2 = f_2 \cdot F_s$ = 2 kHz, the analog signal recovered is $y_a(t) = 13\cos 2000\pi t - 5\sin 4000\pi t$

Quantization

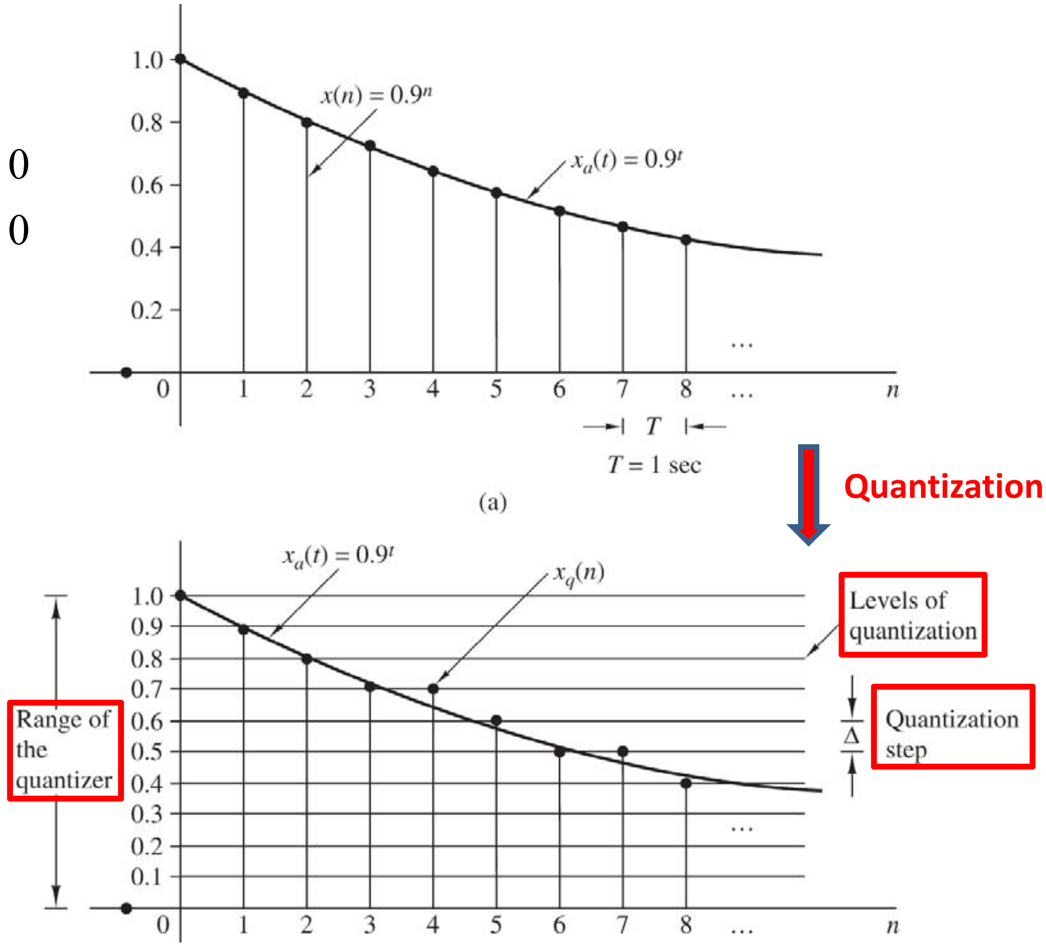
Quantization of Continuous-Amplitude Signals

- **Quantizer operation on the samples $x[n]$:**

$$x_q[n] = Q(x[n])$$

- **Quantization error:** $e_q[n] = x_q[n] - x[n]$

$$x[n] = \begin{cases} 0.9^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Quantization of Continuous-Amplitude Signals (cont'd)

n	$x(n)$ Discrete-time signal	$x_q(n)$ (Truncation)	$x_q(n)$ (Rounding)	$e_q(n) = x_q(n) - x(n)$ (Rounding)
0	1	1.0	1.0	0.0
1	0.9	0.9	0.9	0.0
2	0.81	0.8	0.8	−0.01
3	0.729	0.7	0.7	−0.029
4	0.6561	0.6	0.7	0.0439
5	0.59049	0.5	0.6	0.00951
6	0.531441	0.5	0.5	−0.031441
7	0.4782969	0.4	0.5	0.0217031
8	0.43046721	0.4	0.4	−0.03046721
9	0.387420489	0.3	0.4	0.012579511

↑

$x(n)$ requires n significant digits in order to be represented in full

Quantization of Continuous-Amplitude Signals (cont'd)

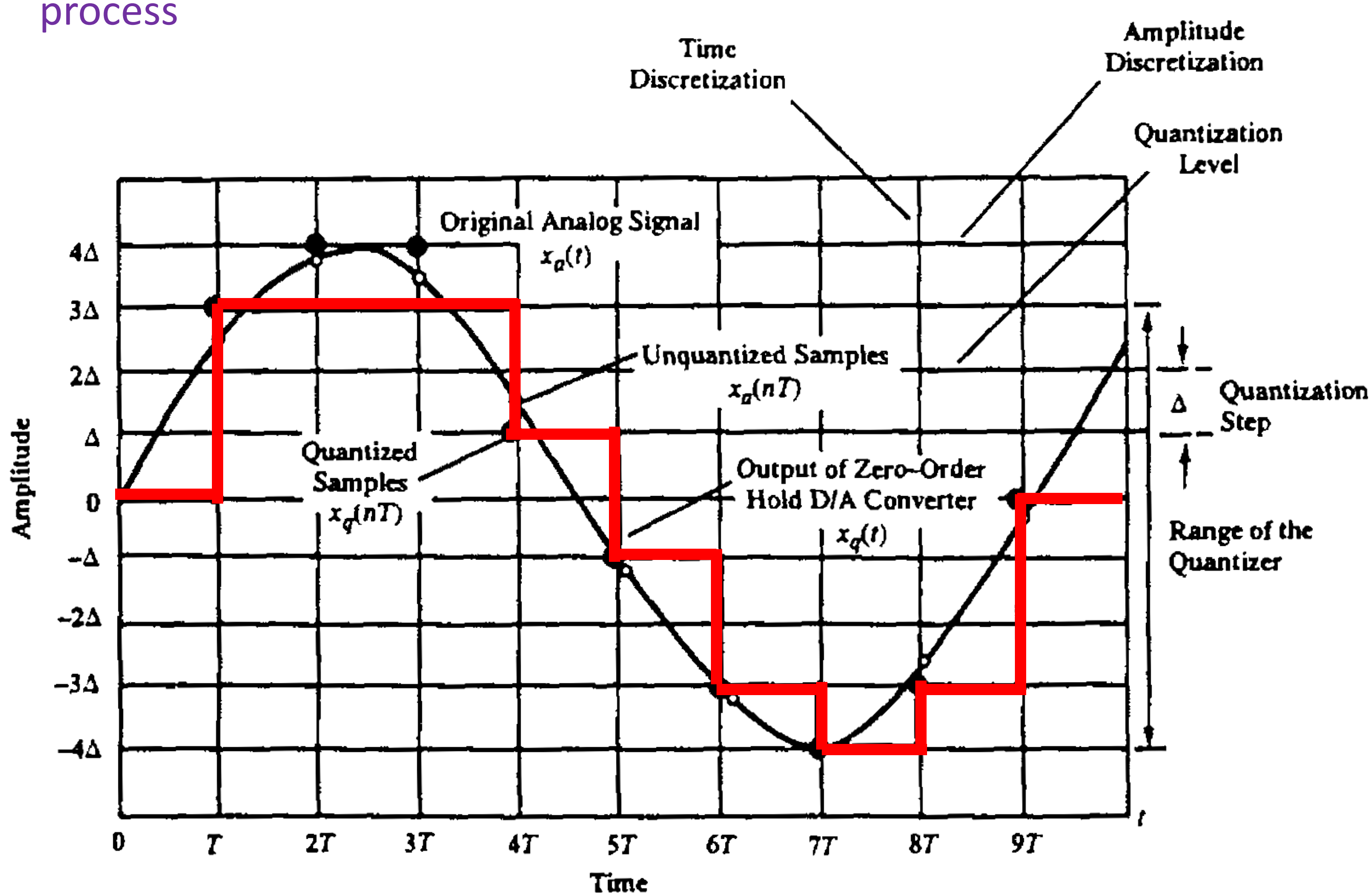
- **Assume want to retain *d* significant digits**
  - Quantization by rounding: drop excess digits and round to nearest digit
  - Quantization by truncation : drop excess digits
- **In previous example, retain only one significant digit**
- **Quantization levels *L*: Number of allowed values of a digital signal**
- **Quantization step Δ**
- **Quantization error for rounding is limited to the range**

$$-\frac{\Delta}{2} \leq e_q[n] \leq +\frac{\Delta}{2}$$

- ***x*[*n*]:**
  - x_{max}: maximum value of $x(n)$
  - x_{min}: minimum value of $\boldsymbol{x(n)}$
  - Dynamic range: $x_{max} - x_{min}$
  - Quantization step is determined by *L* and dynamic range: $\Delta = \frac{x_{\max} - x_{\min}}{L-1}$
- **For a fixed dynamic range, increasing L decreases Δ.**
- **Quantization is an irreversible process**
  - The map $Q(\cdot)$ is a many-to-one mapping since all samples within $\Delta/2$ are assigned the same quantization level

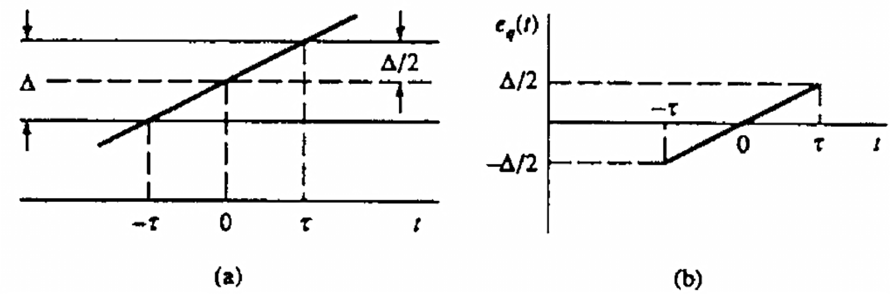
Quantization of Sinusoidal Signals

- $x_a(t) = A \cos\Omega_0 t$
  - If F_s satisfies Nyquist rate, then quantization is the only error in the A/D process



Quantization of Sinusoidal Signals (cont'd)

- **Thus we can evaluate quantization error by quantizing the analog signal $x_a(t)$ instead of the DT signal $x[n]$.**
- **From figure, $x_a(t)$ is almost linear between quantization levels:**



- **τ: time that $x_a(t)$ stays within the quantization levels**
- **Approximate quantization error as:** $e_q(t) = \left(\frac{\Delta}{2\tau}\right)t, \quad -\tau \le t \le \tau$
- **Mean square error power P_q:**

$$P_q = \frac{1}{2\tau}\int_{-\tau}^{\tau} e_q^2(t)\,dt = \frac{1}{2\tau}\int_0^{2\tau}\left(\frac{\Delta}{2\tau}\right)^2 t^2 dt = \frac{\Delta^2}{12}$$

- **If the quantizer uses *b* bits of accuracy and it covers the entire range 2*A*, the quantization step is** $\Delta = \frac{2A}{2^b}$ **and hence**

$$P_q = \frac{A^2/3}{2^{2b}}$$

Quantization of Sinusoidal Signals (cont'd)

- **Average power of the signal $x_a(t) = A\cos\Omega_0 t$:**

$$P_x = \frac{1}{T_p}\int_0^{T_p}\left(A\cos\Omega_0 t\right)^2 dt = \frac{A^2}{2}, \quad \Omega_0 = 2\pi F_0 = \frac{2\pi}{T_p}$$

- **Quality of an A/D converter is measured by the *signal-to-quantization noise ratio* (SQNR)**

$$SQNR \triangleq \frac{P_x}{P_q} = \frac{3}{2}2^{2b}$$

- **Expressed in decibels (dB):**

$$SQNR_{dB} = 10\log_{10}\left(SQNR\right) = 1.76 + 6.02b$$

- **SQNR increases by almost 6 dB for every bit added to the word length**

- **Note: The assumption was that the input signal we a sinusoid and the range of the quantizer spans the dynamic range of the signal. A similar result holds for every such signal, not necessarily a sinusoid, as will be shown later in the course**

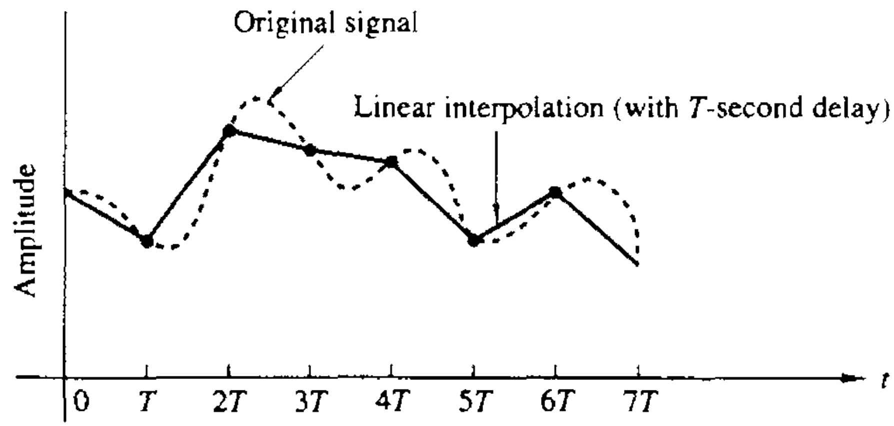
- ***Example*: Most compact disc players use a sampling frequency of 44.1 kHz and 16-bit sample resolution, which implies an SQNR of 96 dB.**

Coding of Quantized Bits

- **Coding process assigns a unique binary number to each quant. Level**
- **For *L* levels, we need *L* different binary numbers**
- **With a word length of *b* bits we can create 2^b levels**
  - Hence $b \geq \log_2 L$
- **Commercially available A/D converters may be obtained with a finite precision of 16 bits or less**
- **Generally, the higher the sampling speed and the finer the quantization, the more expensive the device becomes**

D/A Conversion

- **Ideally, we should use the "sinc" interpolation function**
  - Too complicated for practical purposes
  - Use sub-optimum interpolation techniques
- **Can use *zero-order hold*: Hold constant the value of 1 sample until the next one**
- **Linear interpolation: Connect successive samples with straight line segments**
  - Other higher order techniques



- **Generally, sub-optimum interpolations result in passing frequencies above $F_s/2$**
- **These undesirable frequencies must be removed by passing the output of the interpolator through a proper *analog* filter, called a post-filter or smoothing filter.**