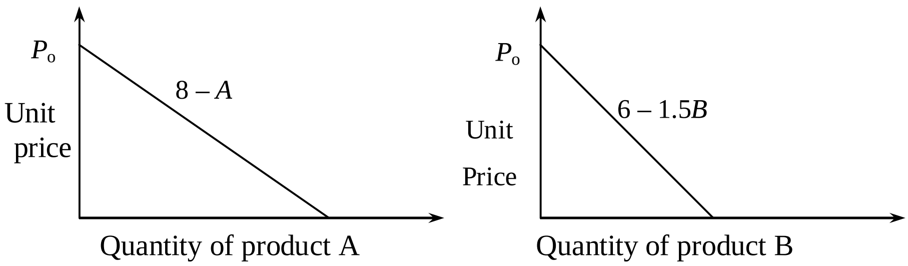
Homework # 2 - Solution

CIVE 646 - Water Resources Systems: Planning and Management
(Fall 2011-12)

4-24 An industrial firm makes two products, *A* and *B*. These products require water and other resources. Water is the scarce resource-they have plenty of other needed resources. The products they make are unique, and hence they can set the unit price of each product at any value they want to. However experience tells them that the higher the unit price for a product, the less amount of that product they will sell. The relationship between unit price and quantity that can be sold is given by the following two *demand* functions:

$P_0(A) = 8 - A$
$P_0(B) = 6 - 1.5B$



(a) What are the amounts of *A* and *B*, and their unit prices, that maximize the total revenue that can be obtained? Formulate the problem and use optimization tool of Matlab to solve it.

(b) Suppose the total amount of *A* and *B* could not exceed some amount T^{max}. What are the amounts of *A* and *B*, and their unit prices, that maximize total revenue, if:

i) $T^{max} = 10$

ii) $T^{max} = 5$

Formulate the problem and use optimization tool of Matlab to solve it.

Water is needed to make each unit of *A* and *B*. The production functions relating the amount of water X_A needed to make *A*, and the amount of water X_B needed to make *B* are $A = 0.5\ X_A$, and $B = 0.25\ X_B$, respectively.

(c) Find the amounts of *A* and *B* and their unit prices that maximize total revenue assuming the total amount of water available is 10 units. Formulate the problem and use optimization tool of Matlab to solve it.

Solution:

(a) Total Revenue Functions:

$TR_A = (8 - A)A = 8A - A^2$
$TR_B = (6 - 1.5B)B = 6B - 1.5B^2$
Maximize TR = $TR_A + TR_A$

Optimization tool of Matlab:

```
function f = objfun(x)
a = x(1); b = x(2);
f = -8*a + a^2 - 6*b + 1.5*b^2;
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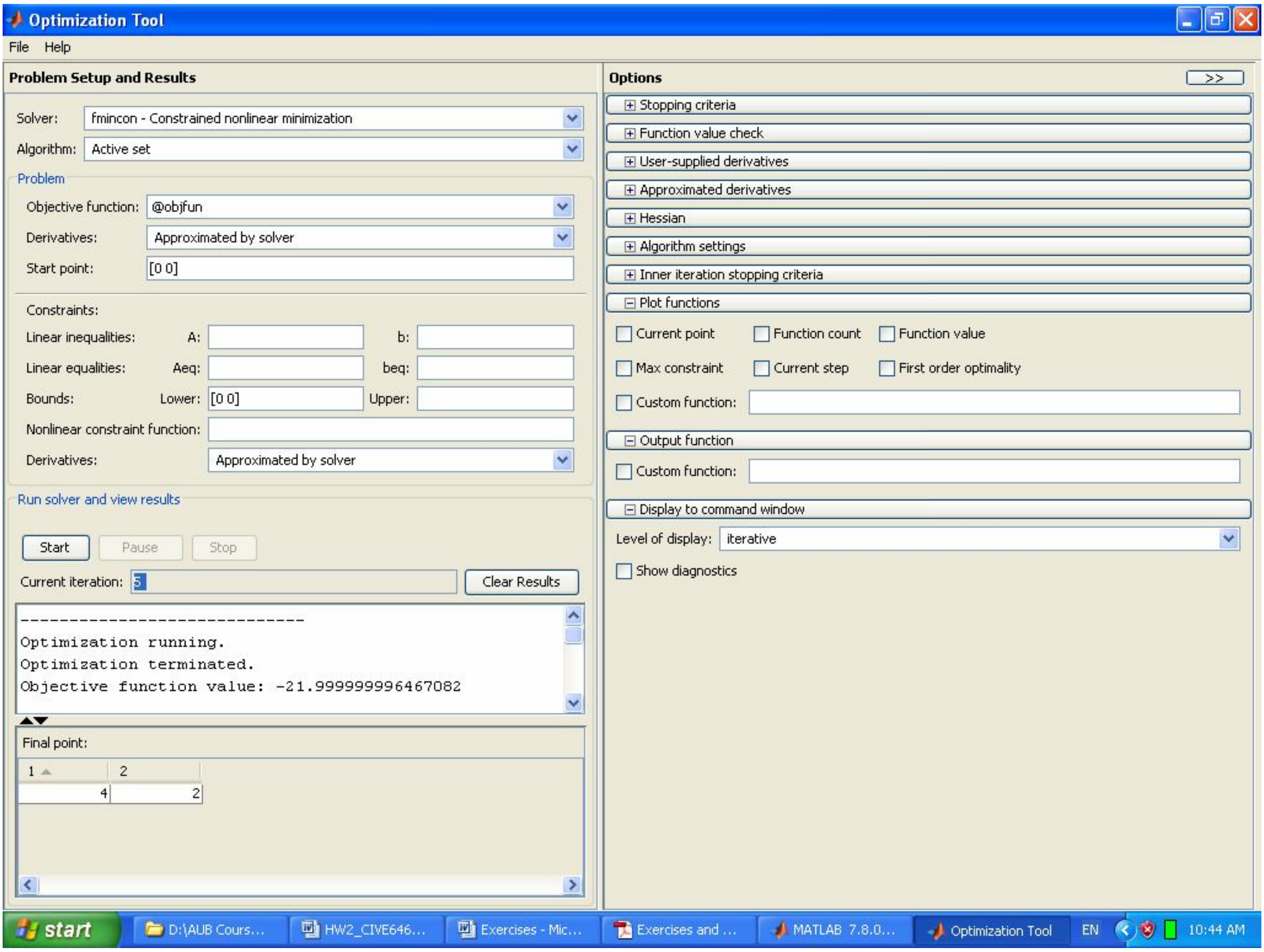


A = 4, P_0(A) = 4
B = 2, P_0(B) = 3
TR = 4*4 + 2*3 = 22

b) In this case:

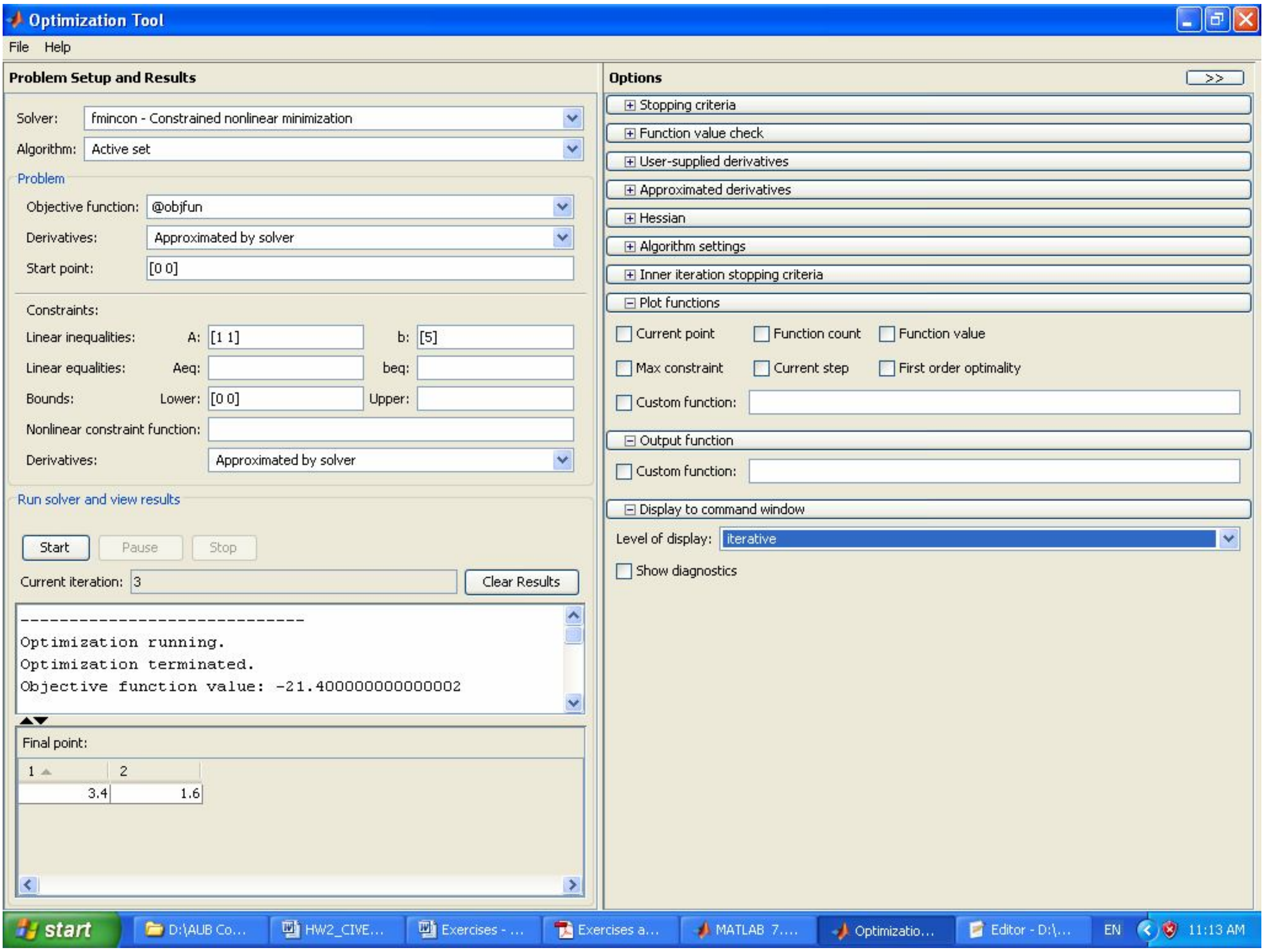
A + B ≤ T^{max}. ≤ 10

The solution for *A* and *B* would be same as (a) since A + B = 6 ≤ 10

A + B ≤ T^{max}. ≤ 5

```
function f = objfun(x)
a = x(1); b = x(2);
f = -8*a + a^2 - 6*b + 1.5*b^2;
```

Subject to A + B ≤ 5



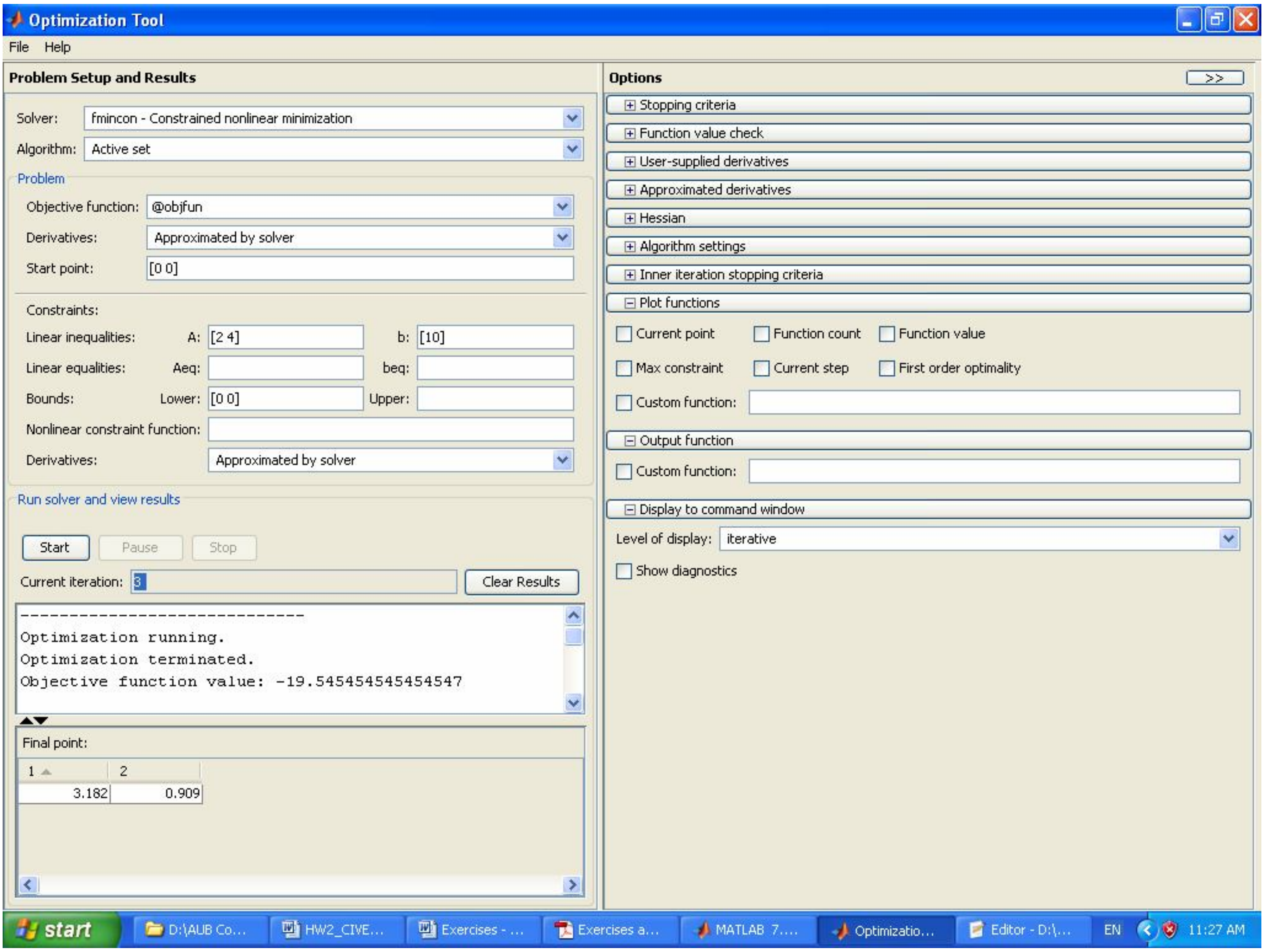
$A = 3$, $P_0(A) = 5$
$B = 2$, $P_0(B) = 3$
$TR = 5*3 + 2*3 = 21$

c) In this case:

$A = 0.5X_A \Rightarrow X_A = 2A$
$B = 0.25X_B \Rightarrow X_B = 4B$
$X_A + X_B \leq 10 \Rightarrow 2A + 4B \leq 10$

```
function f = objfun(x)
a = x(1); b = x(2);
f = -8*a + a^2 - 6*b + 1.5*b^2;
```

Subject to: $2A + 4B \leq 10$

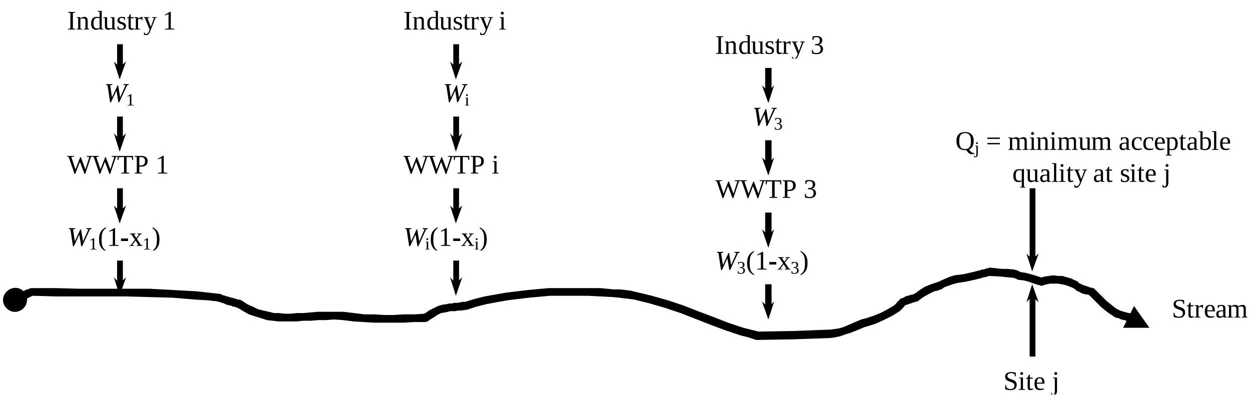


A = 3, $P_0(A)$ = 5 & X_A = 6
B = 1, $P_0(B)$ = 4.5 & X_B = 4
TR = 5*3 + 4.5*1 = 19.5

4.27 Assume that there are *m* industries or municipalities adjacent to a river, which discharge their wastes into the river. Denote the discharge sites by the subscript *i* and let W_i be the kg of waste discharged into the river each day at those sites *i*. To improve the quality downstream, wastewater treatment plants may be required at each site *i*. Let x_i be the fraction of waste removed by treatment at each site *i*. Develop a model for estimating how much waste is removal is required at each site to maintain acceptable water quality in the river at a minimum total cost. Use the following additional notation:

a_{ij} = decrease in quality at site *j* per unit of waste discharged at site *i*
q_j = quality at site j that would result if all controlled upstream discharges were eliminated (i.e., $W_1 = W_2 = 0$)
Q_j = minimum acceptable quality at site *j*
C_i = cost per unit (fraction) of waste removed at site *i*

Solution:

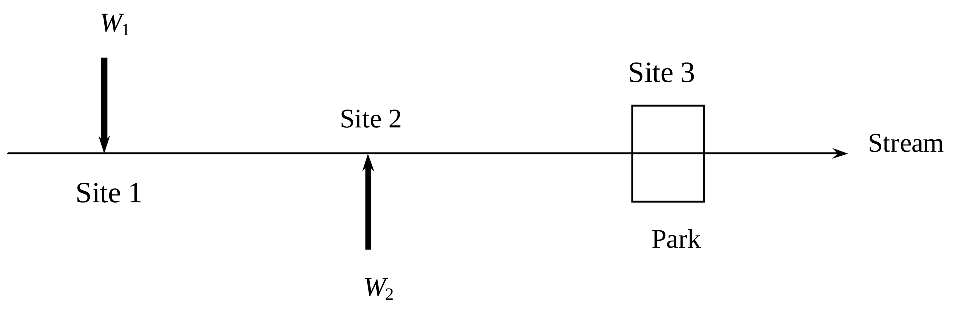


Minimize $\sum_{i=1}^{m} C_i(x_i)$

Subject to $q_j - \sum_{i=1}^{m} a_{ij} w_i (1 - x_i) \geq Q_j \quad \forall j$

$0 \leq x_i \leq 1 \quad \forall i$

4.28 Assume that there are two sites along a stream, i = 1, 2, at which waste (BOD) is discharged. Currently, **without any wastewater treatment, the quality, q_2 and q_3**, at each of sites 2 and 3 is less than the minimum desired, Q_2 and Q_3, respectively.
For each unit of waste removed at site i upstream of site j, **the quality improves by A_{ij}**. How much treatment is required at sites 1 and 2 that meets the standards at a minimum total cost?



Following are the necessary data:

C_i = cost per unit fraction of waste treatment at site i (both C_1 and C_2 are unknown but for the same amount of treatment, whatever that amount, $C_1 > C_2$)
R_i = decision variables, unknown waste removal fractions at sites i = 1, 2

$A_{12} = 1/20$	$W_1 = 100$	$Q_2 = 6$
$A_{13} = 1/40$	$W_2 = 75$	$Q_3 = 4$
$A_{23} = 1/30$	$q_2 = 3$	$q_3 = 1$

Solution:

Minimize $\sum_{i=1}^{2} C_i(R_i)$

subject to: $q_j + \sum_{i=1}^{2} A_{ij} W_i R_i \geq Q_j \quad j = 2, 3$

$0 \leq R_i \leq 1 \quad i = 1, 2$

Minimize $C_1R_1 + C_2R_2$

Subject to $q_2 + A_{12}W_1R_1 \geq Q_2$

$q_3 + A_{13}W_1R_1 + A_{23}W_2R_2 \geq Q_3$

or

Minimize $C_1R_1 + C_2R_2$

Subject to

(at $j = 2$)

$3 + (1/20)(100)\, R_1 \geq 6$ or $R_1 \geq 3/5 = 0.6$

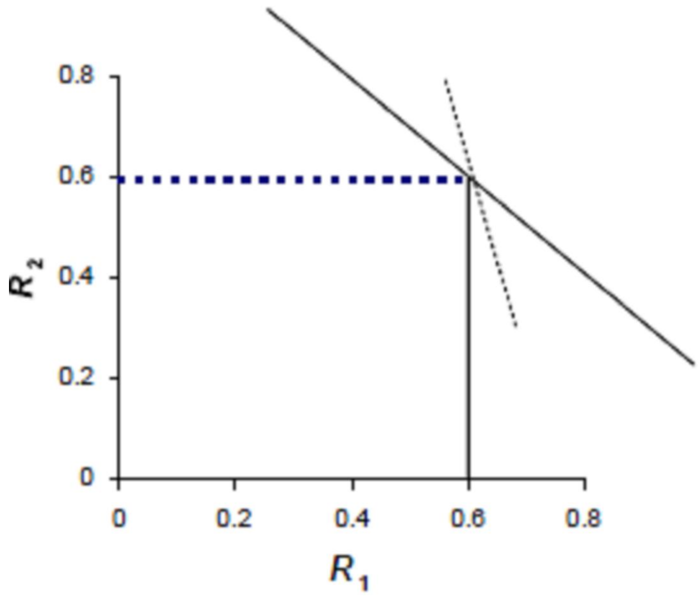
(at $j = 3$)

$1 + (1/40)(100)\, R_1 + (1/30)(75)R_2 \geq 4$ or

$R_1 + R_2 \geq 3/2.5 = 1.2$

$0 \leq R_1 \leq 1$

$0 \leq R_2 \leq 1$



Hence if $C_1 > C_2$, $R_1 = R_2 = 0.6$

4.40 Consider a crop production problem involving three types of crops. How many hectares of each crop should be planted to maximize total income?

Resources:	Max Limits		Resource requirements			
		Crops:	Corn	Wheat	Oats	
Water	1000/week		3.0	1.0	1.5	units/week/ha
Labor	300/week		0.8	0.2	0.3	person hrs/week/ha
Land	625 hectares					
Yield $/ha			400	200	250	

Formulate a linear programming model?.

Solution:

Maximize 400*Corn + 200*Wheat + 250*Oats

Subject to: Water constraint: 3*Corn + Wheat + 105*Oats ≤ 1000
Labor constraint: 0.8*Corn + 0.2*Wheat + 0.3*Oats ≤ 300
Land constraint: Corn + Wheat + Oats ≤ 625

4.43 In Indonesia there exists a wet season followed by a dry season each year. In one area of Indonesia all farmers within an irrigation district plant and grow rice during the wet season. This crop brings the farmer the largest income per hectare; thus they would all prefer to continue growing rice during the dry season. However, there is insufficient water during the dry season to irrigate all 5000 hectares of available irrigable land for rice production. Assume an available irrigation water supply of 32×10^6 m^3 at the beginning of each dry season, and a minimum requirement of 7000 m^3/ha for rice and 1800 m^3/ha for the second crop.

(a) What proportion of the 5000 hectares should the irrigation district manager allocate for rice during the dry season each year, provided that all available hectares must be given sufficient water for rice or the second crop?

(b) Suppose that crop production functions are available for the two crops, indicating the increase in yield per hectare per m^3 of additional water, up to 10,000 m^3/ha for the second crop. Develop a model in which the water allocation per hectare, as well as the hectares allocated to each crop, is to be determined, assuming a specified price or return per unit of yield of each crop. Under what conditions would the solution of this model be the same as in part (a)?

Solution:

a) Let X_R be the hectares of rice to be grown during the dry season and 5000 - X_R be the hectares of the second crop. The total amount of water required is:

$$7000 * X_R + 1800(5000 - X_R) = 32 \text{ x } 10^6$$

Hence X_R = 4435 ha of rice.

b) Let the known parameters:

P_R, P_S equal the price per unit yield of rice and the second crop

Define as the unknown variables:

X_R, X_S = the hectares of rice and second crop
W_R, W_S = the additional water allocations per ha.

Assuming the objective is to maximize total income, the model can be written:

Maximize $P_R\,(7000 + W_R)\,X_R + P_S(1800 + W_S)\,X_S$

Subject to
Water availability: $(7000 + W_R)\,X_R + (1800 + W_S)\,X_S \leq 32 \times 10^6$
Land availability: $X_R + X_S \leq 5000$
Bounds $W_S \leq 10,000\ m^3/ha$

4.45 In Algeria there are two distinct cropping intensities, depending upon the availability of water. Consider a single crop that can be grown under intensive rotation or extensive rotation on a total of *A* hectares. Assume that the annual water requirements for the intensive rotation policy are 16000 m^3 per hectare, and for the extensive rotation policy they 4000 m^3 per hectare. The annual net production returns are 4000 and 2000 dinars, respectively. If the total water available is 320,000 m^3, show that as the available land area *A* increases, the rotation policy that maximizes total net income changes from one that is totally intensive to one that is increasingly extensive.

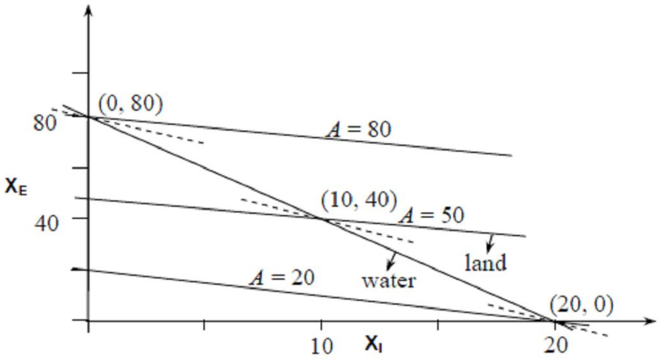
Solution:

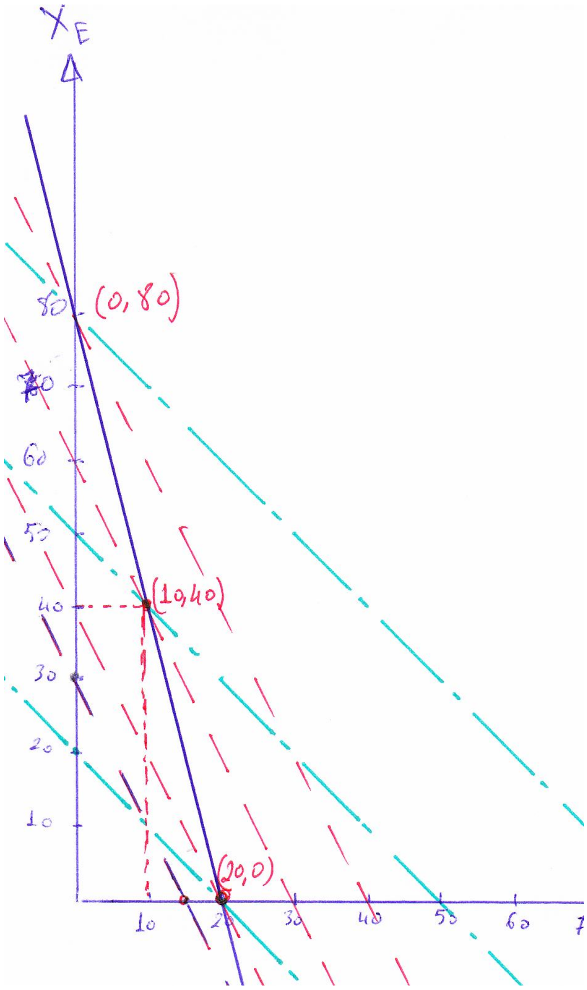
Let X_I be the hectares of intensive cropping intensity and X_E be the hectares of extensive cropping intensity. Maximizing net annual production:

Maximize $4000\,X_I + 2000\,X_E$

Subject to
$16\,X_I + 4\,X_E \leq 320$
$X_I + X_E \leq A$
$X_I, X_E \geq 0$

Solving graphically:

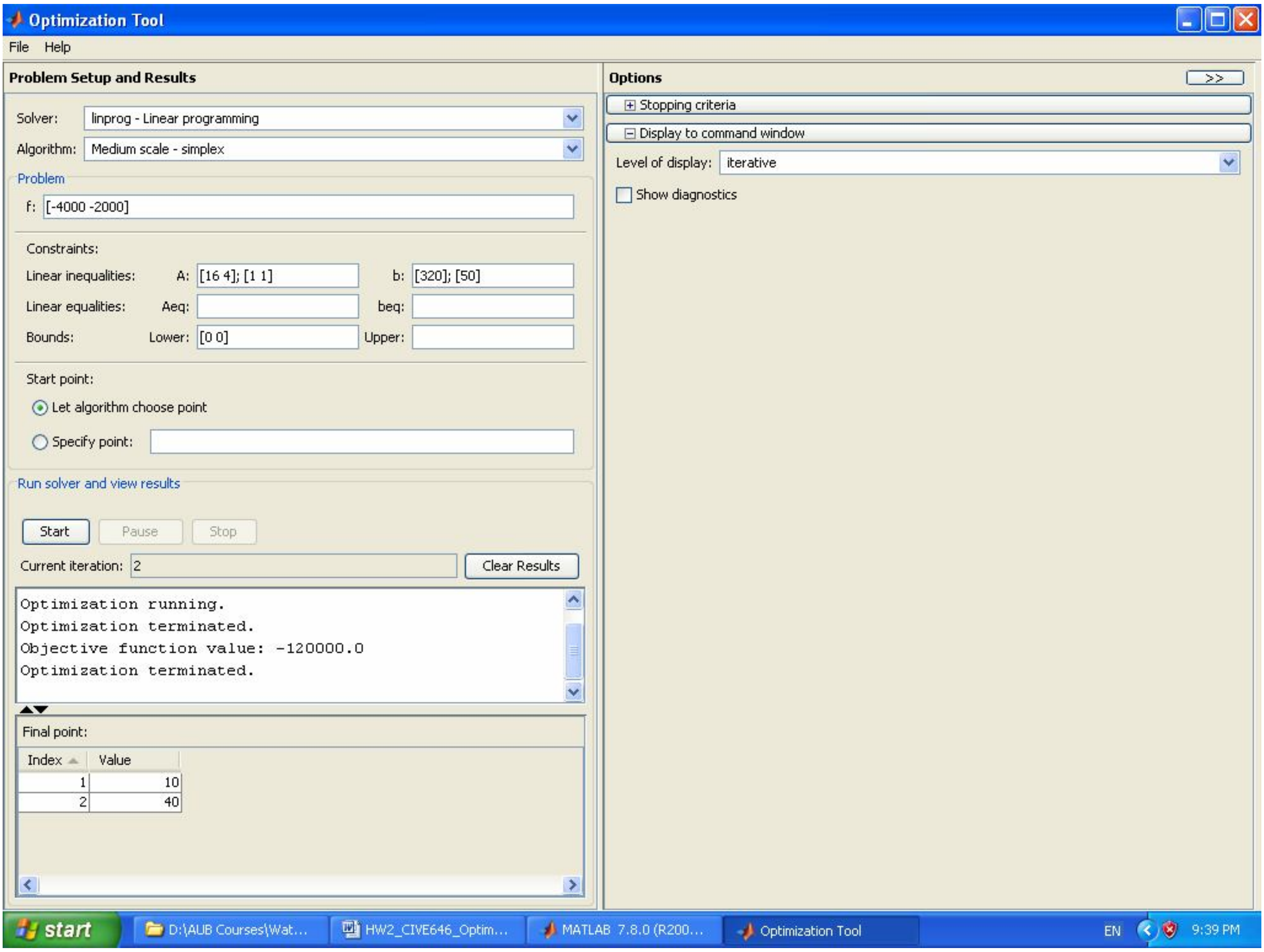




This graph shows that as *A* increases, the optimum value of X_I decreases and X_E increases.

<u>Optimization tool of Matlab:</u>

For A = 50:
$X_I = 10$, $X_E = 40$


Optimization Tool
File Help
Problem Setup and Results
Solver: linprog - Linear programming
Algorithm: Medium scale - simplex
Problem
f: [-4000 -2000]
Constraints:
Linear inequalities: A: [16 4]; [1 1] b: [320]; [50]
Linear equalities: Aeq: beq:
Bounds: Lower: [0 0] Upper:
Start point:
Let algorithm choose point
Specify point:
Run solver and view results
Start Pause Stop
Current iteration: 2 Clear Results
Optimization running.
Optimization terminated.
Objective function value: -120000.0
Optimization terminated.
Final point:
Index Value
1 10
2 40
Options
Stopping criteria
Display to command window
Level of display: iterative
Show diagnostics
start
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