Math 210 - Quiz 1

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Exercise 1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers which is bounded from above and non-decreasing (i.e. $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$). Show that $(a_n)_{n\in\mathbb{N}}$ converges to $l := \sup\{a_n; n \in \mathbb{N}\}$.

Exercise 2. For each of the following subsets of $\mathbb{R}$, compute its infimum. Precise if it is a minimum or not.
All your claims should be proved rigorously but concisely. In this exercise, you are ***not*** *allowed to use the sequential criterion for the infimum or supremum.*

1. $A = \{\frac{1}{n} - \frac{1}{m}; n, m \in \mathbb{N}^*\}$.
2. $B = \{|x - \sqrt{2}|; x \in \mathbb{Q}\}$

Exercise 3. Let $(a_n)_{n\geq 1}$ be the sequence defined for every integer $n \geq 1$ by:

$$a_n = \begin{cases} 1 - \frac{1}{n} & \text{if } n \text{ is even} \\ -2 + \frac{10}{n} & \text{if } n \text{ is odd} \end{cases}$$

Write, using quantifiers, what does it mean for a sequence of real numbers not to be of Cauchy.
Then show that $(a_n)_{n\geq 1}$ diverges **using Cauchy criterion** (Be rigorous).

Exercise 4. Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$. Assume that one can find some sequence $(a_n)_{n\in\mathbb{N}}$ in A such that $a_n \underset{n\to+\infty}{\longrightarrow} x$. Compare $\sup A$ and x (state your claim then prove it).

Exercise 5. Prove that if a sequence $(a_n)_{n\in\mathbb{N}}$ has no convergent subsequence in $\mathbb{R}$, then $|a_n| \underset{n\to+\infty}{\longrightarrow} +\infty$.

Exercise 6. (The Stolz-Cesaro Theorem)

1. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers. Assume that $b_n > 0$ for every $n \geq 1$ and that $b_1 + \cdots + b_n \underset{n\to+\infty}{\longrightarrow} +\infty$.

   Show that
$$\frac{a_n}{b_n} \underset{n\to+\infty}{\longrightarrow} l \in \mathbb{R} \implies \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \underset{n\to+\infty}{\longrightarrow} l.$$

2. Deduce that if $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are two sequences of real numbers such that $(B_n)_{n\geq 1}$ is increasing and tending to $+\infty$, then
$$\frac{A_{n+1} - A_n}{B_{n+1} - B_n} \underset{n\to+\infty}{\longrightarrow} l \in \mathbb{R} \implies \frac{A_n}{B_n} \underset{n\to+\infty}{\longrightarrow} l.$$

3. (Vague question, Bonus) The result stated in Question 2 can be seen as a "discrete" version of a classical result in Calculus. Can you guess and explain which one?

4. Deduce that if $a_n \underset{n\to+\infty}{\longrightarrow} l \in \mathbb{R}$, then $\frac{a_1+\cdots+a_n}{n} \underset{n\to+\infty}{\longrightarrow} l$.

5. <u>Application:</u>

   (a) **Use Question 2** to show that the following limit exists (and compute it):
$$\lim_{n\to+\infty} \frac{\sum_{k=1}^{n} \frac{1}{\sqrt{k}}}{\sqrt{n}}.$$

   (b) More generally, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that $\sqrt{n}x_n \underset{n\to+\infty}{\longrightarrow} C \in \mathbb{R}$. Show that the following limit exists (and compute it)
$$\lim_{n\to+\infty} \frac{\sum_{k=1}^{n} x_k}{\sqrt{n}}.$$

Solution

Exercise 1. Denote by $S \subseteq \mathbb{R}$ the range of the sequence, i.e. $S = \{x_n; n \in \mathbb{N}\}$, and by l its supremum (which is finite as S is bounded from above by the given and $\mathbb{R}$ has the least upper bound property). Let $\epsilon > 0$. Since $l - \epsilon < l$ and l is the least upper bound of S, we can find some $s \in S$ such that $s > l - \epsilon$. Since $s \in S$, we can write $s = a_{n_0}$ for some $n_0 = n_0(\epsilon) \in \mathbb{N}$. But $(a_n)_{n\in\mathbb{N}}$ is non-decreasing. Hence, for every $n \geq n_0$, $a_n \geq a_{n_0}$. By transitivity of the relation $\leq$, we deduce that $a_n > l - \epsilon$. Since l is an upper bound of S, we deduce that

$$\forall n \geq n_0, l - \epsilon < a_n \leq l < l + \epsilon.$$

This proves that $(a_n)_{n\in\mathbb{N}}$ converges to l.

Exercise 2.

1. We claim that $\inf A = -1$ and that it is not a minimum. Indeed, let $x \in A$. Then, there exists some $n, m \geq 1$ such that $x = \frac{1}{n} - \frac{1}{m}$. Since $m \geq 1$, we deduce that $\frac{1}{m} \leq 1$. Since $\frac{1}{n} > 0$, we deduce that $x > -1$. This being true for every $x \in A$, we deduce that -1 is an upper bound for A. Moreover, let $\epsilon > 0$. By the Archimedean property, there exists some $n_0 \in \mathbb{N}^*$ such that $\frac{1}{n_0} < \epsilon$. In particular, $z := \frac{1}{n_0} - 1 < -1 + \epsilon$. Obviously, $z \in A$ (take $n = n_0$ and $m = 1$). Hence z is an element in A greater than $1 - \epsilon$. Hence $-1 + \epsilon$ is not a lower bound for A. This being true for every $\epsilon > 0$, we deduce that $\inf A = -1$ indeed.
   Finally, -1 is not a minimum for A as we checked earlier that $x > -1$ for every $x \in A$.

2. We claim that $\inf B = 0$. Indeed, 0 is clearly a lower bound for B in $\mathbb{R}$. Moreover, take $\epsilon > 0$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can find a rational number x_0 such that $\sqrt{2} - \epsilon < x_0 < \sqrt{2} + \epsilon$. In other terms, $|x_0 - \sqrt{2}| < \epsilon$. Clearly, $z := |x_0 - \sqrt{2}| \in B$. Hence ϵ is not a lower bound for B. This being true for every $\epsilon > 0$, we deduce that $\inf B = 0$ indeed.
   Now 0 is not a minimum. Indeed $0 \in B$, if and only if, there exists some $x \in \mathbb{Q}$ such that $|x - \sqrt{2}| = 0$, i.e. $x = \sqrt{2}$. Since $\sqrt{2}$ is an irrational number, we deduce that $0 \notin B$.

Exercise 3.

1. A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is not of Cauchy in $\mathbb{R}$, if and only, if

$$\exists \epsilon_0 > 0; \forall n \in \mathbb{N}, \exists k \geq n, \exists l \geq n; |x_k - x_l| \geq \epsilon_0.$$

2. Consider now the sequence $(x_n)_{n\geq 1}$ given by the exercise. For every $n \geq 1$, one has:

$$x_{n+1} - x_n = \begin{cases} -3 + \frac{1}{n} + \frac{10}{n+1} & \text{if } n \text{ is even} \\ 3 - \frac{1}{n+1} - \frac{10}{n} & \text{if } n \text{ is odd} \end{cases}$$

   If $n \geq 11$,

$$\max\left\{\frac{1}{n} + \frac{10}{n+1}, \frac{1}{n+1} + \frac{10}{n}\right\} \leq \frac{1}{n} + \frac{10}{n} = \frac{11}{n} \leq 1.$$

   Hence, if $n \geq 11$,

$$|-3 + \frac{1}{n} + \frac{10}{n+1}| = |3 - (\frac{1}{n} + \frac{10}{n+1})| = 3 - (\frac{1}{n} + \frac{10}{n+1}) \geq 2,$$

   and similarly

$$|3 - \frac{1}{n+1} - \frac{10}{n}| \geq 2.$$

   Hence,

$$\forall n \geq 11, |x_{n+1} - x_n| \geq 2. \tag{1}$$

   Put $\epsilon_0 := 2$ and consider an arbitrary $n \geq 1$. Take $k := \max\{11, n\}$ and $l := k + 1$. We have $k, l \geq n$ and, by (1),

$$|x_k - x_l| \geq \epsilon_0.$$

   Hence the sequence $(x_n)_{n\geq 1}$ is not of Cauchy. A fortiori, it diverges.

Exercise 4. Let A be a subset of $\mathbb{R}$ and $x \in \mathbb{R}$. Assume that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of elements in A such that $a_n \underset{n\to+\infty}{\longrightarrow} x$. We claim that $\sup A \geq x$. Indeed, one can assume without loss of generality that A is bounded from above so that $\sup A \in \mathbb{R}$ (otherwise the result is trivial). Arguing by contradiction, assume that $x > \sup A$. Put $\epsilon_0 := \frac{x-\sup A}{2} > 0$. Since the sequence $(a_n)_{n\in\mathbb{N}}$ converges to x, then there exists $n_0 \in \mathbb{N}$ such that $a_{n_0} > x - \epsilon$. But $x - \epsilon > x - (x - \sup A) = \sup A$. Hence $a_{n_0} > \sup A$. This leads to a contradiction as $a_{n_0} \in A$ and $\sup A$ is an upper bound for A. We conclude that $\sup A \geq x$.

Remark on Exercise 5. You cannot deduce the result of Exercise 5. immediately from the contrapositive of Bolzano-Weierstrass theorem. Indeed, the latter says that a bounded sequence has a convergence subsequence; hence a sequence that does not have a convergent subsequence is unbounded. But an unbounded sequence need to tend to $+\infty$ (even in absolute value). It is true that the result of this exercise is closely related to Bolzano-Weierstrass but one should be careful in the reasoning. Actually, once you deduce from Bolzano-Weierstrass theorem that your sequence is unbounded, in order to conclude that it tends to $+\infty$ in absolute value; you should use once again Bolzano-Weierstrass (and not just say that it follows from the unboundness of your sequence). A concise proof is given below.

Exercise 5. Consider a sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers having no convergent subsequences in $\mathbb{R}$. Arguing by contradiction, suppose that the sequence $(|a_n|)_{n\in\mathbb{N}}$ does not tend to $+\infty$. The key point is to observe that this is equivalent to saying that there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ that is bounded. This fact follows immediately from the negation of the statement "$(|a_n|)_{n\in\mathbb{N}}$ tend to $+\infty$').
Now we can use Bolzano-Weierstrass theorem for the new sequence $(a_{n_k})_{k\in\mathbb{N}}$, to obtain a convergent subsequence $(a_{n_{k_l}})_{l\in\mathbb{N}}$. The latter is also a subsequence of the original sequence $(a_n)_{n\in\mathbb{N}}$. Hence, by our given, it cannot converge. Contradiction. In consequence, $|a_n| \underset{n\to+\infty}{\longrightarrow} +\infty$.

Exercise 6. 1. For every $n \geq 1$, let $B_n := b_1 + \cdots + b_n$. Recall that $b_n > 0$ for every $n \geq 1$ and that $B_n \underset{n\to+\infty}{\longrightarrow} +\infty$. Let $\epsilon > 0$. Since $\frac{a_n}{b_n} \underset{n\to+\infty}{\longrightarrow} l$, then there exists some $N \geq 1$ such that for every $n \geq N$, $|\frac{a_n}{b_n} - l| < \frac{\epsilon}{2}$, i.e.

$$\forall n \geq N, \, |a_n - b_n l| < b_n \frac{\epsilon}{2}. \tag{2}$$

Take now an arbitrary $n \geq N$. We have

$$\begin{aligned}
\Big|\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} - l\Big| &= \frac{\big|(a_1 + \cdots + a_n) - (lb_1 + \cdots + lb_n)\big|)}{b_1 + \cdots + b_n} \\
&= \frac{\big|\sum_{k=1}^{n} (a_k - b_k l)\big|}{B_n} \\
&\leq \frac{\sum_{k=1}^{n} |a_k - b_k l|}{B_n} \qquad (3) \\
&= \frac{\sum_{k=1}^{n_0-1} |a_k - b_k l|}{B_n} + \frac{\sum_{k=n_0}^{n} |a_k - b_k l|}{B_n} \\
&< \frac{\sum_{k=1}^{n_0-1} |a_k - b_k l|}{B_n} + \frac{\epsilon}{2} \frac{\sum_{k=1}^{n_0-1} b_k}{B_n} \qquad (4) \\
&\leq \frac{\sum_{k=1}^{n_0-1} |a_k - b_k l|}{B_n} + \frac{\epsilon}{2} \qquad (5)
\end{aligned}$$

Inequality (3) comes from the triangular inequality, (4) is due to (2) and (5) comes from the fact that $(B_n)_{n\geq 1}$ is increasing (as the b_i's are positive). Since $B_n \underset{n\to+\infty}{\longrightarrow} +\infty$ (and $\sum_{k=1}^{n_0-1} |a_k - b_k l|$ is a finite quantity independent of n), we deduce that $\frac{\sum_{k=1}^{n_0-1} |a_k - b_k l}{B_n} \underset{n\to+\infty}{\longrightarrow} 0$. Hence

$$\exists n_1 \geq 1; \forall n \geq n_1, 0 \leq \frac{\sum_{k=1}^{n_0-1} |a_k - b_k l}{B_n} < \frac{\epsilon}{2}.$$

Hence

$$\forall n \geq \max\{n_0, n_1\}, \, \Big|\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} - l\Big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows that $\frac{a_1+\cdots+a_n}{b_1+\cdots+b_n} \underset{n\to+\infty}{\longrightarrow} l$.

2. Let $(A_n)_n$ and $(B_n)_n$ be sequences such that $(B_n)_n$ is increasing and tending to $+\infty$. Construct sequences $(a_n)_n$ and $(b_n)_n$ as follows:

$$a_n = \begin{cases} A_1 & \text{if } n = 1 \\ A_n - A_{n-1} & \text{if } n \geq 2 \end{cases}, \quad b_n = \begin{cases} B_1 & \text{if } n = 1 \\ B_n - B_{n-1} & \text{if } n \geq 2 \end{cases}$$

Notice that for every $n \geq 1$,

$$a_1 + \cdots a_n = A_n \text{ and } b_1 + \cdots b_n = B_n.$$

Observe also that the assumptions on $(B_n)_n$ imply that $b_n > 0$ for every $n \geq 1$ and that

$$b_1 + \cdots b_n = B_n \underset{n\to+\infty}{\longrightarrow} +\infty.$$

It suffices to apply now the previous questions.

3. L'Hospital's Rule for evaluating limits of quotient in indeterminate forms!

4. Apply Question 1 for $(a_n)_n$ the same sequence and $b_n := 1$, $n \geq 1$. The sequence $(b_n)_n$ is indeed positive and satisfies $b_1 + b_2 + \cdots + b_n = n \underset{n\to+\infty}{\longrightarrow} +\infty$.

5. (a) Let $A_n = \sum_{k=1}^n \frac{1}{\sqrt{k}}$ and $B_n = \sqrt{n}$. The conditions on $(B_n)_n$ are clearly fulfilled. Moreover, for every $n \geq 1$,

$$\frac{A_{n+1} - A_n}{B_{n+1} - B_n} = \frac{\frac{1}{\sqrt{n+1}}}{\sqrt{n+1} - \sqrt{n}} = 1 + \sqrt{\frac{n}{n+1}}.$$

Hence $\frac{A_{n+1}-A_n}{B_{n+1}-B_n} \underset{n\to+\infty}{\longrightarrow} 2$. By Question 2, we deduce that $\frac{A_n}{B_n} \underset{n\to+\infty}{\longrightarrow} 2$, i.e.

$$\frac{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}}{\sqrt{n}} \underset{n\to+\infty}{\longrightarrow} 2.$$

(b) More generally, consider a sequence $(x_n)_n$ such that $\sqrt{n}x_n \longrightarrow C \in \mathbb{R}$. Take $A_n := x_1 + \cdots + x_n$ and $B_n = \sqrt{n}$.

We have that:

$$\frac{A_{n+1} - A_n}{B_{n+1} - B_n} = (\sqrt{n+1} + \sqrt{n})x_{n+1} = \left(1 + \sqrt{1 + \frac{n}{n+1}}\right)\sqrt{n+1}x_{n+1} \underset{n\to+\infty}{\longrightarrow} 2C.$$

By Question 2, we deduce that $\frac{A_n}{B_n} \underset{n\to+\infty}{\longrightarrow} 2$, i.e.

$$\frac{\sum_{k=1}^n x_k}{\sqrt{n}} \underset{n\to+\infty}{\longrightarrow} 2C.$$