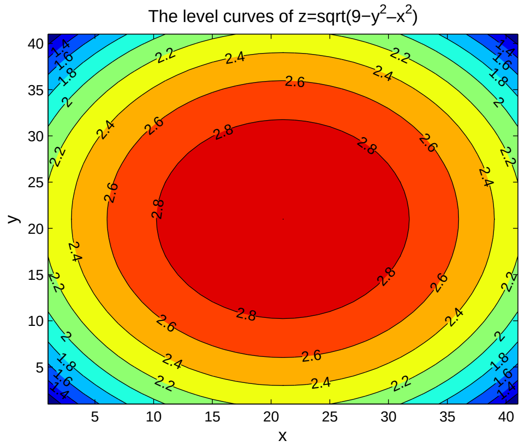
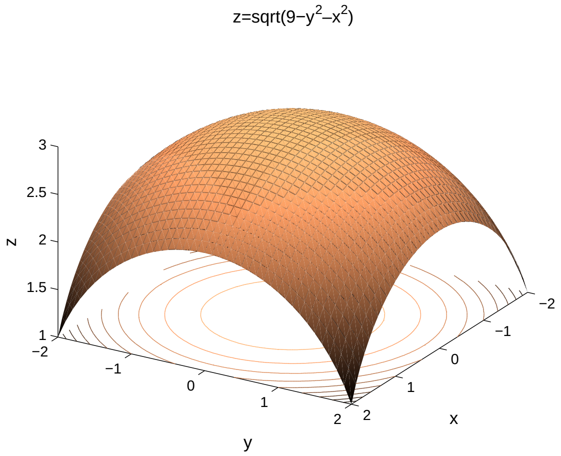
1) Let $f(x,y) = \sqrt{9 - x^2 - y^2}$

a) Give the domain and the range of f: $D_f = \{(x,y) \in \mathbb{R}^2; x^2+y^2 \leq 9\}$, and $Range(f) = [0,3]$

b) Sketch some level curves of f



c) Sketch the graph of f



d) find $f_x(0,0)$ and $f_y(0,0)$ (by using the definition)

$f_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} \frac{\sqrt{9-x^2}-3}{x} = \lim_{x\to 0} \frac{(9-x^2)-9}{x(\sqrt{9-x^2}+3)} = -\lim_{x\to 0} \frac{x}{\sqrt{9-x^2}+3} = 0$, and $f_y(0,0) = 0$ (by symmetry)

e) prove that f is differentiable at $(0,0)$

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)-f(0,0)-f_x(0,0)\Delta x - f_y(0,0)\Delta y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{9-x^2-y^2}-3}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(9-x^2-y^2)-9}{\sqrt{x^2+y^2}(\sqrt{9-x^2-y^2}+3)} \text{ (multiplying by the conjugate)}$$

$$= -\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{\sqrt{9-x^2-y^2}+3} = 0$$

hence f is differentiable at $(0,0)$

2) Find the directional derivative of the $f(x,y) = 2xy - y^2$ at $P = (5,5)$ in the direction of $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$

<u>solution</u>:

$\nabla f = (2y, 2x - 2y) \Rightarrow \nabla f(5,5) = (10,0)$; $\mathbf{v} = \frac{\mathbf{u}}{|u|} = \frac{4\mathbf{i}+3\mathbf{j}}{\sqrt{4^2+3^2}} = \frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{j}$; hence

$(Df)_{\mathbf{u},P} = \nabla f(P) \bullet \mathbf{v} = 8$

3) Find the direction of maximum increase of $f(x,y) = x^2 - 3xy + 4y^2$ at $P(1,2)$. Is there is a direction $\mathbf{u}$ in which the rate of change of f at $P(1,2)$ equals 14? Justify your answer.

<u>solution</u>:

$\nabla f = (2x - 3y, -3x + 8y)$, then $\nabla f(1,2) = (-4, 13)$, and $|\nabla f(1,2)| = \sqrt{16+169} = \sqrt{185}$

The direction of maximum increase of f is the direction of $\nabla f(1,2)$, hence $(-4/\sqrt{185}, 13/\sqrt{185})$

The rate of change in the direction of maximum increase is $\sqrt{185} < 14$, then there is no direction $\mathbf{u}$ in which the rate of change of f at $P(1,2)$ equals 14.

4) The derivative of of $f(x,y)$ at $P(1,2)$ in the direction of $\mathbf{u} = \mathbf{i} + \mathbf{j}$ is $2\sqrt{2}$ and in direction of $\mathbf{v} = -2\mathbf{j}$ is -3. Find the derivative of f in the direction of $\mathbf{w} = -\mathbf{i} - 2\mathbf{j}$.

<u>solution</u>:

$(Df)_{\mathbf{u},P} = 2\sqrt{2} \Rightarrow f_x(P) \times \frac{1}{\sqrt{2}} + f_y(P) \times \frac{1}{\sqrt{2}} = 2\sqrt{2} \Rightarrow f_x(P) + f_y(P) = 4$

$(Df)_{\mathbf{v},P} = -3 \Rightarrow -f_y(P) = -3 \Rightarrow f_y(P) = 3$, and then $f_x(P) = 1$, hence

$(Df)_{\mathbf{w},P} = 1 \times \frac{-1}{\sqrt{5}} + 3 \times \frac{-2}{\sqrt{5}} = \frac{-7}{\sqrt{5}}$

5) The derivative of of $f(x,y,z)$ at P is greatest in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. In this direction, the value of the derivative is $2\sqrt{3}$. Find ∇f at P. Find the derivative of f at P in the direction of $\mathbf{w} = \mathbf{i} + \mathbf{j}$.

<u>solution</u>:

- $\nabla f = k.\frac{\mathbf{v}}{|\mathbf{v}|}$ (cause ∇f and $\mathbf{v}$ are collinear). Moreover $k = 2\sqrt{3}$, and then $\nabla f = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
- $(Df)_{\mathbf{w},P} = \nabla f(P) \bullet \frac{\mathbf{w}}{|\mathbf{w}|} = 2 \times \frac{1}{\sqrt{2}} + 2 \times \frac{1}{\sqrt{2}} + 2 \times 0$

6) Given the surface $z = x^2 - 4xy + y^3 + 4y - 2$ containing the point $P(1,-1,-2)$.

- Find an equation of the tangent plane to the surface at P.
- Find an equation of the normal line to the surface at P.

<u>solution</u>:

Let $w(x,y,z) = x^2 - 4xy + y^3 + 4y - 2 - z$; $\nabla w = (2x - 4y, -4x + 3y^2 + 4, -1) \Rightarrow \nabla w(1,-1,-2) = (6,3,-1)$, and the

- $(T): 6(x-1) + 3(y+1) - (z+2) = 0$
- $(N): x = 6t + 1, y = 3t - 1, z = -t - 2\ , \quad t \in \mathbb{R}$

7) Find parametric equation for the line tangent to the curve of intersection of the surfaces $xyz = 1$ and $x^2 + 2y^2 + 3z^2 = 6$ at the point $P(1, 1, 1)$.

<u>solution</u>:

$\nabla f = (yz, xz, xy) \Rightarrow \nabla f(1, 1, 1) = (1, 1, 1)$, and $\nabla g = (2x, 4y, 6z) \Rightarrow \nabla g(1, 1, 1) = (2, 4, 6)$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 4 & 6 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} \Rightarrow \text{tangent line } T : x = 1 + 2t, y1 - 4t, z = 1 + 2t$$

8) By about how much will $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$ change if the point $p(x, y, z)$ moves from $P(3, 4, 12)$ a distance of 0.1 unit in the direction of $\mathbf{u} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$?

<u>solution</u>:

the change df is given by $df = (Df)_{\mathbf{u},P} \times ds$; $\nabla f = (\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2})$, and $\nabla f(P) = (\frac{3}{169}, \frac{4}{169}, \frac{12}{169})$

$(Df)_{\mathbf{u},P} = \nabla f(P) \bullet \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{3}{169} \times \frac{3}{7} + \frac{4}{169} \times \frac{6}{7} - \frac{12}{169} \times \frac{2}{7} = \frac{9}{1183}$; then $df = 0.9/1183 \simeq 0.0008$

9) Locate all relative extrema and saddle points of $f(x, y) = x^3 - y^3 - 2xy + 6$.

<u>solution</u>:

critical points: $(0, 0)$ (saddle point); $(-2/3, 2/3)$ (local maximum) and the maximum is $f(-2/3, 2/3) = 120/27$.

10) Locate all relative extrema and saddle points of $f(x, y) = 4xy - x^4 - y^4$.

<u>solution</u>:

The critical points are $(0, 0)$ (saddle point); $(1, 1)$ (local maximum) and $f(1, 1) = 2$; $(-1, -1)$ (local maximum) and $f(-1, -1) = 2$.

11) Find the absolute minimum and maximum values of $f(x, y) = 3xy - 6x - 3y + 7$ on the closed triangular region R whose vertices are $(0, 0), (3, 0)$ and $(0, 5)$.

<u>solution</u>:

<u>critical point:</u> $\partial f/\partial x = 3y - 3$ and $\partial f/\partial y = -3x - 6$, then the only critical point is $(-2, 1)$

<u>on the path $x = 0$:</u> $g(y) = f(0, y) = -3y + 7$, $g'(y) = -3 \neq 0$,

<u>on the path $y = 0$:</u> $h(x) = f(x, 0) = -6x + 7$, $h'(x) = -6 \neq 0$,

<u>on the path $y = 5 - \frac{5}{3}x$:</u> $k(x) = f(x, x) = -5x^2 + 14x - 8$, $k'(x) = -10x + 14$, $k'(x) = 0 \rightarrow x = 7/5$, and then $y = 8/3$

point	value
(-2,1)	9
(0,0)	7
(3,0)	-11
(0,5)	-8
(7/5,8/3)	81/5

The minimum is then equals to -11 at $(3, 0)$; the maximum is equals to $81/5$ at $(7/5, 8/3)$

12) Find the absolute minimum and maximum of the function $f(x,y) = x^2 + 2y^2 - y - 1$ over the region $R = \{(x,y);\ x^2 + y^2 \leq 1\ ,\ y \geq 0\}$.

solution:

critical point: $\partial f/\partial x = 2x$ and $\partial f/\partial y = 4y - 1$, then the only critical point is $(0, 1/4)$

on the path $y = 0$: $g(x) = f(x, 0) = x^2 - 1$, $g'(x) = 2x$; $g'(x) = 0 \Rightarrow x = 0$, and then $y = 0$; $(0, 0)$ is also a critical point

on the semi circle $x^2 + y^2 = 1 \Rightarrow x^2 = 1 - y^2$:

$h(y) = f(1 - y^2, y) = y^2 - y$, $h'(y) = 2y - 1$, $h'(y) = 0 \rightarrow y = 1/2$, and then $x = \pm\sqrt{3}/2$

point	value
(0,1/4)	-9/8
(-1,0)	0
(1,0)	0
(0,0)	-1
$(\sqrt{3}/2, 1/2)$	1/2
$(-\sqrt{3}/2, 1/2)$	1/2

The minimum is then equals to $-9/8$ at $(0, 1/4)$; the maximum is equals to $1/2$ at $(\sqrt{3}/2, 1/2)$ and $(-\sqrt{3}/2, 1/2)$.