Math. 271, Final Examination, January 29, 1997

Problem 1. **18 pts.**
Which topics of the course did you like the most ? the least ? Write about those topics and about your reasons.

Problem 2. **18 pts.**
Write what you know about equipotent sets and cardinal numbers.

Problem 3. **12 pts.**
Write the definitions of:
(a) a well-ordered set;
(b) an ultrafilter;
(c) a δ-multiplicative family of sets;
(d) Lim inf A_n when $A_0, A_1, A_2, \ldots$ are sets.

Problem 4. **12 pts.**
Write the statements of:
(a) the Cantor diagonal theorem;
(b) the Zermelo theorem;
(c) the Baire theorem;
(d) the Souslin hypothesis.

Problem 5. **10 pts.**
Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be given by the formula $f(m, n) = m \cdot n$.
(a) Prove that f is a surjection and that f is not an injection.
(b) Let $A = \{0, 2, 4, \ldots\}$ denote the set of all even natural numbers. Find $f^{-1}(A)$ and $f(A \times A)$.

Problem 6. **10 pts.**
Let X be a set, $\mathcal{A}$ be a σ-algebra of subsets of X, and μ be a measure on $\mathcal{A}$.
(a) Prove that $\mu(B \cup C) = \mu(B) + \mu(C) - \mu(B \cap C)$ for every $B, C \in \mathcal{A}$.
(b) Let $\mathcal{F} = \{B \in \mathcal{A} : \mu(B) > 0\}$. Is $\mathcal{F}$ a filter? Why?

Problem 7. **10 pts.**
Let C denote the set of all sequences $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} a_n = 0$ and $a_n \in \mathbb{Q}$ for each n. What is the cardinality of the set C ? Justify your answer!

Problem 8. **20 pts.**
Let $A_0, A_1, A_2, \ldots$ and $B_0, B_1, B_2, \ldots$ be sets.
(a) Which inclusion does always hold between the sets $\bigcap_{n=0}^{\infty}(A_n \cup B_n)$ and $\left(\bigcap_{n=0}^{\infty} A_n\right) \cup \left(\bigcap_{n=0}^{\infty} B_n\right)$? Prove your claim.
(b) Give an example when the inclusion in (a) is proper.
(c) Show that the two sets in (a) are equal when $A_0 \supset A_1 \supset A_2 \supset \ldots$ and $B_0 \supset B_1 \supset B_2 \supset \ldots$
(d) What can be said about the two sets in (a) when $A_0 \subset A_1 \subset A_2 \subset \ldots$ and $B_0 \subset B_1 \subset B_2 \subset \ldots$?

Good Luck!