Math 307–Final Exam (Spring 03) B. Shayya

Problem 1

(20 pts) Let N be a positive integer. We showed in class that the estimate

$$\|f\|_{L^\infty(B(0,R))} \leq \frac{C_N}{|B(0,R)|} \int \frac{|f(x)|}{(1+R^{-2}|x|^2)^N} dx$$

holds for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ with $\operatorname{Supp} \widehat{f} \subset B(0, 1/R)$, where C_N is a constant that depends only on N and n. Is it possible to obtain the stronger estimate

$$\|f\|_{L^\infty(B(0,R))} \leq \frac{C'_N}{|B(0,R)|} \int_{B(0,2R)} |f(x)| dx?$$

Problem 2

(20 pts) Let E be a compact subset of $\mathbb{R}^n$ and let $2 \leq p < \infty$. Suppose there is a measure $\mu \in P(E)$ with $\widehat{\mu} \in L^p(\mathbb{R}^n)$. Prove that $\dim E \geq 2n/p$.

Problem 3

(20 pts) Prove that to every interval $I \subset \mathbb{R}$ and number $B > 0$ there is a function $\phi \in C_0^\infty(\mathbb{R})$ with

$$\|\phi\|_{L^\infty} \leq 1 \qquad \text{and} \qquad \int_I |\widehat{\phi}(\xi)| d\xi > B.$$

Problem 4

(20 pts) Let $1 > \epsilon_1 \geq \epsilon_2 \geq \cdots$ be a decreasing sequence of positive numbers and let E be a Cantor set with dissection ratios $\{\epsilon_k\}$: start with the interval $[0, 1]$, remove the middle ϵ_1 proportion, then remove the middle ϵ_2 proportion of each of the resulting intervals and so forth. Prove that

$$\dim E = \frac{\log 2}{\log \frac{2}{1-\epsilon}}$$

where $\epsilon = \lim_{k\to\infty} \epsilon_k$.

Problem 5

(20 pts) Let

$$E = \{x \in \mathbb{R} : x = \sum_{k=1}^{\infty} \frac{\epsilon_k}{k!}, \ \epsilon_k = 0 \text{ or } 1\}.$$

Prove that $\dim E = 0$.

Problem 6

(20 pts) Let E be a Lebesgue measurable subset of $\mathbb{R}^n$. True or False:

(i) $\dim E = n \ \Rightarrow \ |E| > 0$?

(ii) $\dim E = 0 \ \Rightarrow \ E$ is countable?