Mathematics *341 Final Exam*
Time: 2.5 hrs

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Part 1: (80%) **Answer any 8** problems of the following 10 problems.
(*Include all essential details and omit all trivialities*)

1. Show that an injective module has no proper essential extensions. (Yes: trivial)
2. Use Baer's Lemma to show that, over a p.i.d R, every divisible R-module is injective.
3. Show that every R-module M has an injective resolution: a long exact sequence as $0 \to M \to Q_1 \to Q_2 \to \ldots\ldots\ldots Q_n \to \ldots\ldots$ where each Q_i is an injective R-module.
4. (a) Over **Z**-modules, show that $Z_5 \otimes Q = 0$
   (b) Over (general R-modules), show that $M \otimes R \cong M$
5. Let $0 = a \otimes b \in A \otimes B$ (tensoring of R-modules). Suppose R is a commutative domain, A is torsion-free and B is free. Show that a=0 or b-0.
6. *Let* $M \prec_e Q$ be an essential extension of R-modules over any domain R.
   If M is also divisible & Q is torsion-free, show that M=Q. (Hint: Let $q \in Q - M$)
7. Let M be a divisible R-module over a commutative domain. If M is also torsion-free, show that M is injective. (Hint: In applying Baer's Lemma, start with any non-zero elt. in I).
8. Let M be a torsion module over a p.i.D R. Let $M = \oplus M_p$ (Primary Decomposition Theorem).
   Let A be a submodule of M. Show that $A = \oplus (A \cap M_p)$
9. Show that every injective endomorphism of an Artinian module is an automorphism .
10. Show that the inclusion map $Z \oplus Z \oplus Z \oplus \ldots\ldots\ldots \to Z \times Z \times Z \times \ldots\ldots$ dose not split (in Z-modules)

Part 2: (20%)

A) Every injective Z-module which is also projective must be the zero module.
(Hint: Restate the problem using divisibility).

B) Let $f: I \to M$ be linear map of R-modules. If R is a commutative domain, I is an ideal of R, M is torsion free, $f(i) = im$ & $f(i') = i'm'$ where i and i' are non-zero elts of I. Show that $m=m'$

C) Let $R \subset S$ be an extension of rings such that $1_R = 1_S$. Show that $S^n \cong S \otimes R^n$

D) Let R be a left noetherian ring. Show that any direct sum of injective R-modules is also injective.

E) Let R be a domain with the property that every f.g R-module is projective.
Show that every submodule of a free R-module is free.