First Semester 2003-2004
January 30, 2004
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MATHEMATICS 341
Final Examination

Time: 2 Hours

1. (a) Let D be a principal ideal domain. Prove that every proper prime ideal of D is maximal.
   (b) Let D be an integral domain, $a, b \in D$. Suppose that $a^n = b^n$ and $a^m = b^m$ for two relatively prime positive integers m and n. Prove that $a = b$. (Use the ring (field) of quotients of D)

2. Let R be a left Noetherian (Artinian) ring with identity. Prove that every finitely generated unitary R-module is Noetherian (Artinian).

3. Let M be an R-module and $N \subseteq M$ a submodule of M such that M/N is projective. Prove that
$$M \cong N \oplus (M/N).$$

4. (a) State, without proof, the structure theorem for finitely generated modules over a principal ideal domain.
   (b) Use the above theorem to classify all Z-modules of order 108

5. Let R be an integral domain and let M be a free R-module . Prove that M is torsion-free.

6. Let $f : M \to N$ be an R-module homomorphism and let $i : Ker f \to M$ be the canonical inclusion. Prove that if P is an R-module and $g : P \to M$ is an R-module homomorphism such that $f \circ g = 0$, then there exists a unique R-module homomorphism $h : P \to Ker f$ such that the diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{h} & \downarrow g & & \\ Ker f & \xrightarrow{i} & M & \xrightarrow{f} & N \end{array}$$

is commutative

7. Let J be an <u>injective</u> module over a ring R. Prove that if $\psi : A \to B$ is any R-module monomorphism , then $\overline{\psi} : Hom_R(B, J) \to Hom_R(A, J)$ is an epimorphism of abelian groups. ($\overline{\psi}(f) = f \circ \psi$)

8. (a) Prove that if m and n are relatively prime positive integers, then
$$Z_m \otimes Z_n = \{0\} \quad where \otimes = \otimes_Z$$
   (b) Prove that if Q is the additive group of rationals, then
$$Q \otimes Q \cong Q \quad where \otimes = \otimes_Z$$

9. If R is a nonzero commutative ring with identity such that every submodule of every free R-module is free. Prove that every ideal of R is principal.