Math 341 Final Examination N. Nahlus
Time: 2 hours Jan 27, 2005

Part I (80%) **Answer 1-3 AND any other 5 problems** of the following 10 problems.

1. Over a p.i.d R, show that every submodule of $\boldsymbol{R^n}$ is free of rank at most n.

2. Prove or disprove by a counter example.
a) Over a p.i.d R , a torsion free module is free.
b) If $A \otimes_R B = 0$, then A=0 or B=0.

3. Prove or disprove by a counter example.
a) If $a \otimes_R b = 0$ *(in* $A \otimes_R B$*)*,, A is free and B is torsion free, then a=0 or b=0.
b) If $A \otimes_R B = 0$, $A' \leq A$, *then* $A' \otimes_R B = 0$.

4. Prove or disprove by a counter example.
a) Let $f : A \twoheadrightarrow Z_{36}$ be a surjective morphism of abelian groups, and let $g : Z \to Z_{36}$ be a morphism of abelian groups, then there exists a morphism $h : Z \to A$ such that $f \circ h = g$.
b) Over a domain D, a free module is torsion free.
c) Over an integral domain D, let T be the submodule of torsion elements of a module M, then M/T is torsion free

5. Let $f : M \to M$ be an epimorphism of an Noetherian module M.
Show that f is an isomorphism.

6. Use Zorn's Lemma to show that every proper submodule of a finitely generated R-module is contained in a maximal submodule.

7. Prove or disprove by a counter example.
a) Every submodule of a Noetherian module is also Noetherian.
b) Every subring of a Noetherian ring is also Noetherian.
c) Every Artinian integral domain is a field

8. (i) State very carefully 3 equivalent definitions for a projective module and
(ii) prove any (non-obvious) implication of the equivalence.
(iii) Is **Q** a projective **Z**-module ? Justify

9. Prove or disprove by a counter example.
a) Every submodule of a projective module is projective.
b) Every quotient module of a projective module is projective.
c) If every R-module is injective, then every R-module is projective.

10. Use Nakayama's Lemma, to prove that every finitely generated projective module over a local ring is free.

Part 2. **(20%)** **See next page**

Part 2. **(20%)**

1) Prove one part of the following

(a) Let M be the ring of n by n matrices over a left noetherian ring R. Let S be the subring of M of all the upper triangular matrices.
Show that S is a left noetherian ring.

(b) Show that division rings are the only rings over which every module is free.
(Hint: Recall that R is a division ring iff R has no non-trivial maximal left ideals.)

2) Show that $A \oplus B$ *is projective* $\Leftrightarrow$ *A and B are projective*

3) Let $0 \to A \to B \to C \to 0$ be an exact sequence of modules (over a ring R).
Prove or disprove (by a counter example)
(a) If A and B are projective, then C is projective.
(b) If A and C are projective, then B is projective.
(c) If B and C are projective, then A is projective.