Math 341 — Fall 2005-06
Modules and Rings I
Take-home final exam, due Monday, January 30, at 5pm

in my mailbox at the mathematics department
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Remark: Make sure to solve the problems from both sides of the sheet. This take-home exam is not a problem set: you may not discuss the problems with any person until every student has handed in the exam. You may consult any books or other written references you like for this exam, but the course texts and your lecture notes ought to be sufficient. If you use a result not in the course texts or in your lecture notes, be sure to include a proof of that result, with a reference to the source.

All nine problems are equally weighted, even though they vary in difficulty.

I will be available TTh 3–5 in my office if you are unsure about the meaning of the statements in the problems for this quiz. I will not give hints about how to solve these problems below, **however** I will be happy to answer any questions you may have about the lecture notes, previous problem sets, and course texts.

Good luck!

Problem 1: If R is a Noetherian integral domain, show that every nonzero element of R is either a unit or factors into a product of irreducible elements. (N.B., the factorization need not be unique.)

Problem 2: Let A, B, C be R-modules, and assume given homomorphisms $f : A \to B$ and $g : B \to C$ such that for **every** R-module P, the following sequence is exact:

$$0 \to \mathrm{Hom}_R(C,P) \xrightarrow{g^*} \mathrm{Hom}_R(B,P) \xrightarrow{f^*} \mathrm{Hom}_R(A,P).$$

Show that the sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ must be exact.

Hints: (i) to show that g is surjective, take $P = \operatorname{coker} g$, (ii) to show that $gf = 0$, take $P = C$, (iii) to show that $\ker g \subset \operatorname{Image} f$, take $P = \operatorname{coker} f$.

Problem 3: Let R be a ring in which every ideal is projective (this is a weaker condition than being a PID). Show (by imitating the corresponding proof for PIDs) that every submodule of the free module R^n is isomorphic to a direct sum $I_1 \oplus I_2 \oplus \cdots \oplus I_k$ of ideals $I_1, \ldots, I_k$ with $k \leq n$.

Problem 4: Let V be a five-dimensional vector space over $\mathbf{Q}$ with basis $\{e_1, \ldots, e_5\}$. We are given a linear transformation $T : V \to V$ such that

$$\begin{aligned} T(e_1) &= e_4, \\ T(e_2) &= 0, \\ T(e_3) &= e_2 - e_3 - 2e_4 - e_5, \\ T(e_4) &= e_3 + e_5, \\ T(e_5) &= e_3 - 2e_4 + e_5. \end{aligned}$$

a) Write the matrix of T with respect to the given basis, and use it to show that the characteristic polynomial $p_T(\lambda)$ is equal to $\lambda^5 + 4\lambda^3$.

b) Find a $\mathbf{Q}$-basis for each of the two (nonzero) primary components of V, viewed as a module over $R = \mathbf{Q}[\lambda]$ in the usual way. (The first component corresponds to the irreducible polynomial λ, and the second corresponds to the irreducible polynomial $\lambda^2 + 4$.)

c) Decompose each primary component into a direct sum of primary cyclic modules, and use this to find the **primary** rational canonical form of T. (This is the rational canonical form as used in Hungerford, not Jacobson.)

d) Find the Jordan canonical form of T.

Problem 5: Let M be a module over a ring R.

a) Show that there exist free modules F and G, and there exist homomorphisms α and β, so that we have an exact sequence $G \xrightarrow{\alpha} F \xrightarrow{\beta} M \to 0$.

b) If R is a Noetherian ring, and M is a finitely-generated R-module, show that we can take F and G to be finitely-generated free modules.

c) Assume we are given another module M' and a homomorphism $\mu : M \to M'$. Assume we also know an exact sequence $G' \xrightarrow{\alpha'} F' \xrightarrow{\beta'} M' \to 0$. (In applications, F' and G' are usually free modules, but we do not need this.) Show that there exist homomorphisms φ, γ such that the following diagram commutes:

$$\begin{array}{ccccccc} G & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & M & \to & 0 \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\mu} & & \\ G' & \xrightarrow{\alpha'} & F' & \xrightarrow{\beta'} & M' & \to & 0 \end{array}$$

Hint: Free modules are projective.

Problem 6: We work over the polynomial ring $R = \mathbf{Q}[\lambda]$. Consider the 2×2 matrix

$$A = \begin{pmatrix} \lambda^2 + 1 & \lambda^3 + \lambda^2 \\ \lambda^3 - \lambda^2 & \lambda^4 - \lambda^2 \end{pmatrix} \in M_2(R).$$

Find (invertible!) matrices $P, Q \in GL_2(R)$ such that PAQ is diagonal.

Problem 7: Find a triangular basis for each of the following free $\mathbf{Z}$-modules:

a) $M = \langle (3,4,1), (4,5,2), (1,9,6) \rangle \subset \mathbf{Z}^3$,

b) $M = \{(a,b,c) \in \mathbf{Z}^3 \mid 7a + 8b + 9c = 0\}$,

c) $M = \{(x,y) \in \mathbf{Z}^2 \mid x + 3y \equiv 0 \bmod 8 \text{ AND } x - 4y \equiv 0 \bmod 9\}$. Hint: the answer to part (b) is useful.

Problem 8: Let R be a commutative ring, let $I \subset R$ be an ideal, and let M be an R-module. Use the properties of the tensor product that I mentioned in the last lecture (basically Proposition IV.5.4, applied to $0 \to I \to R \to R/I$, and Theorem IV.5.7 of Hungerford) to prove that

$$(R/I) \otimes M \cong M/IM.$$

Note: IM is the submodule of M **generated by** all elements of the form im, for all elements $i \in I, m \in M$. Note that $I \otimes M$ is not the same thing as IM, and part of your answer should involve finding some connection between them.

Problem 9: Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules, where R is any commutative ring. Consider the ideals I', I, I'' given by

$$I' = \operatorname{Ann} M', \qquad I = \operatorname{Ann} M, \qquad I'' = \operatorname{Ann} M''.$$

a) Show that $I'I'' \subset I$. (Recall that $I'I''$ is generated by products of the form $i'i''$, with $i' \in I'$ and $i'' \in I''$.

b) Give two examples of exact sequences (preferably with $R = \mathbf{Z}$) where I', I'' are nonzero, such that in the first example $I'I'' = I$ and $I'I'' \subsetneq I$.

c) Assume that $I' + I'' = R$, i.e., that I' and I'' are relatively prime ideals. Show that our exact sequence must be split.