AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH-CMPS 350
FINAL EXAMINATION-Make-up
FALL 2007-2008
Two hours

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1		
2		
3		
4		
5		
6		
TOTAL		

Let $\Omega \subset \mathbb{R}^2$ be an open connected domain with $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \Phi$ (empty set). Let a, b f be continuous functions ($C(\overline{\Omega})$), such that:

$$a(x, y) \geq a_0 > 0,\ b(x, y) \geq 0,\ \forall (x, y) \in \overline{\Omega}.$$

Consider the problem of finding $u : \overline{\Omega} \to \mathbb{R}$, that verifies:

$$(P) \begin{cases} -div(a(x, y)\nabla u) + b(x, y)u &= f(x, y) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{cases}$$

1. State Green's formula and derive the variational (weak) form of (P) in the form:

   $$\text{(P')}\ u \in V : A(u, v) = F(v),\ \forall v \in V.$$

   Define the Sobolev space $H^1(\Omega)$ and specify the bilinear form $A(.,.)$, the functional $F(.)$, and the subspace V.

2. Prove existence and uniqueness of the solution to (P').

3. Consider the one-dimensional version corresponding to the problem of question 1, i.e. let:
$\Omega = (-1, 1)$; $T = \{v \in H^1(\Omega) | v(-) = v(1) = 0\}$; $A(u, v) = \int_\Omega (a(x)u^{'}v^{'} + b(x)uv)dx$;
$F(v) = \int_\Omega (f(x)v)dx$, where $a(x) \geq a_0 > 0$ and $b(x) \geq 0$.
Consider the variational problem:

$$u \in T: \ A(u, v) = F(v), \forall v \in T. \tag{1}$$

Assume $u \in H^2$ and let

$$S_N = span\{Q_n | n = 0, ..., N\}, N > 2$$

be the orthonormal set of Legendre polynomials in $L^2(\Omega)$. Give an estimate on $||u^{'} - \Pi_{N-1}(u')||$, where $\Pi_{N-1}(u')$ is the L^2 projection of $u^{'}$ on S_{N-1}.

4. By studying the inner product $< u^{'}, Q_0 >$, show that:

$$T_N = span\{\varphi_n | n = 1, ..., N-1\},$$

with:

$$\varphi_n = P_{n+1}(x) - P_{n-1}(x),\ n = 1, 2, ..., N-1.$$

is a suitable subspace of polynomials of degree n for a Galerkin approximation to (1). Give an estimate on $||u - r_N(u)||_1$, where $r_N(u)(x) = \int_{-1}^{x} \Pi_{N-1}(u^{'})$.

5. Let u_N be the Galerkine spectral approximation $u_N \in T_N$

$$u_N(x) = \Sigma_{n=1}^{N} c_n \varphi_n(x).$$

Give the Galerkin formulation based on (1),

$$u_N \in T_N : \tag{2}$$

(a) Give the system of linear equations:

$$Ac = F;\ c \in \mathbb{R}^N,\ A \in \mathbb{R}^{N,N},\ F \in \mathbb{R}^N. \tag{3}$$

equivalent to (2). Specify the matrix A the vectors F and c. Give (with proof) the properties of A.

(b) Prove the estimation:

$$||u - u_N||_1 \leq c \min_{v \in T_N} ||u - v||_1$$

6. Let A be an $n \times n$ inverible matrix. Let $x^0 \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, $r^0 = b - Ax^0$, and x the unique solution to $Ax = b$. Prove that:

$$x \in x^0 + span\{r^0, Ar^0, ..., A^{n-1}r^0\}.$$