

Physics Department

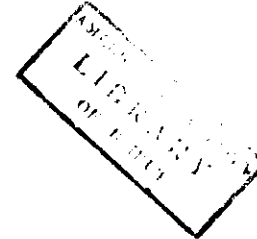
Physics 305
Final Exam

Feb. 4, 1997
Time: 3 hours

Name: _____

I.D. No. _____

Please try to work out all questions



Content	Grade
1. Time Evolution of quantum state	_____
2. Time-Independent perturbation Theory . . .	_____
3. Variational Method	_____
4. Scattering	_____

Total:



(1) Time Evolution of quantum states

The wave function of a rigid rotator with a Hamiltonian

$$\hat{H} = \hat{L}^2 / 2I, \quad (I = \text{moment of inertia})$$

is given by

$$\langle \theta, \phi | \psi(0) \rangle = \sqrt{\frac{3}{4\pi}} \sin \theta \sin \phi \quad (1)$$

- What is $\langle \theta, \phi | \psi(t) \rangle$?
- If L_z is measured, what are its values, and what is the probability of measuring these values?
- Calculate $\langle L_x \rangle$ for this state $|\psi(t)\rangle$.
- Suppose that the rigid rotator is placed in a uniform magnetic field $\vec{B} = (0, 0, B_0)$ and \hat{H} is given:

$$\hat{H} = \frac{\hat{L}^2}{2I} + \omega_0 \hat{L}_z, \quad (\omega_0 = \text{constant})$$

If $\langle \theta, \phi | \psi(0) \rangle$ is as given in Equation (1) above, what is $\langle \theta, \phi | \psi(t) \rangle$?

- Find $\langle L_x \rangle$ at time t .

Hints:

to (a): note that $\sin \phi = \frac{(e^{i\phi} - e^{-i\phi})}{2i}$. Express Eq. (1) in terms of the spherical harmonics Y_{lm} .

to (c): you may use $L_x = \frac{(L_+ + L_-)}{2i}$, where L_+ and L_- are the raising and lowering operators, respectively.

(2) Time-independent Perturbation

The spin Hamiltonian of a spin - 1/2 particle is given by

$$\hat{H} = \mu_s \vec{S} \cdot \vec{B}$$

Suppose, a particle is placed in an external magnetic field

$$\vec{B} = (0, B_2, B_0),$$

where $B_0 \gg B_2$.

- (a) First, determine the energy eigenvalues of \vec{H} exactly.
- (b) Secondly, apply perturbation theory to calculate the energy eigenvalues through second order in (B_2/B_0) , and compare the results with the exact values obtained in part (a). Do they agree?

Hint: in part (a), during your trip, define $\omega_0 = -\mu_s B_0$ and $\omega_2 = \mu_s B_2$

(3) Variational Method

The Hamiltonian of a one-dimensional oscillator of mass m is given by

$$\hat{H} = \frac{\hat{P}_x^2}{2m} + b \hat{x}^4, \quad (b > 0)$$

The exact value of the ground-state energy is

$$E_0 = 1.06 b^{1/3} \left(\frac{\hbar^2}{2m} \right)^{2/3}$$

Apply the variational method to estimate the ground-state energy of the oscillator, and to see how close this result will be to the exact one.

Hint: You need to work with the formalism in chapter 7, in particular:

$$\begin{aligned} \hat{a}^+ |n\rangle &= \sqrt{n+1} |n+1\rangle \\ \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle \\ \langle n' | \hat{a}^+ |n\rangle &= \sqrt{n+1} \delta_{n', n+1} \\ \langle n' | \hat{a} |n\rangle &= \sqrt{n} \delta_{n', n-1} \\ \hat{a} |0\rangle &= 0 \end{aligned}$$

Express \hat{x} and \hat{P}_x in terms of \hat{a}^+ and \hat{a} .

(4) Scattering Theory

- (a) For a short-range potential, verify that outside the range of the potential, the wave function

$$U(r, \theta) = \frac{1}{r} \left(1 + \frac{i}{kr} \right) e^{ikr} \cos \theta$$

represents an outgoing P-wave ($\ell = 1$).

- (b) Consider a beam of particles represented by a plane wave e^{ikz} , scattered by an impenetrable sphere of radius a , and assume $ka \ll 1$. Consider only S and P components in the scattered wave (d-waves excluded).

Show that to the order of $(ka)^2$, the differential cross section at an angle θ is:

$$\frac{d\sigma}{d\Omega} = a^2 \left\{ 1 - \frac{1}{3} (ka)^2 + 2 (ka)^2 \cos \theta \right\}$$

Hints:

In part (a): the term "outside" means $V = 0$. Ask yourself: which equation should be satisfied by the radial part of the wave function.

Note that $P_1(\cos \theta) = \cos \theta$

In part (b): somewhat tricky.

First obtain a general expression of the scattering amplitude which will contain two constants.

Study the behavior of the wave function at $r \rightarrow a$ to determine the constants.

Note that excluding the d-waves contribution means that $\cos^2 \theta = 1/3$ (average over all direction).