Physics Department

Physics 305
Final Exam

Feb. 4, 1997
Time: 3 hours

Name:______________________________

I.D. No. ___________________________

Please try to work out all questions

Content | Grade

1. Time Evolution of quantum state .... _______
2. Time-Independent perturbation Theory . . . _______
3. Variational Method .... _______
4. Scattering .... _______

Total:

(1) Time Evolution of quantum states

The wave function of a rigid rotator with a Hamiltonian

$$\hat{H} = \hat{\vec{L}}^2 / 2I, \quad (I = \text{moment of inertia})$$

is given by

$$<\theta, \phi \,|\psi\,(o) = \sqrt{\frac{3}{4\pi}} \quad \sin\theta \sin\phi \qquad (1)$$

a) What is $<\theta, \phi \,|\psi\,(t)>$?

b) If L_z is measured, what are its values, and what is the probability of measuring these values

c) Calculate $< L_x >$ for this state $|\psi\,(t)>$.

d) Suppose that the rigid rotator is placed in a uniform magnetic field $\vec{B} = (0,0, B_o)$ and $\hat{H}$ is given:

$$\hat{H} = \frac{\hat{\vec{L}}^2}{2I} + \omega_o \hat{L}_z \quad , \quad (\omega_o = \text{constant})$$

If $<\theta, \phi \,|\psi\,(o)>$ is as given in Equation (1) above, what is $<\theta, \phi \,|\psi\,(t)>$?

e) Find $< L_x >$ at time t .

Hints:

to (a) : note that $\sin\phi = \frac{(e^{i\phi} - e^{-i\phi})}{2i}$. Express Eq. (1) in terms of the spherical harmonics $Y_{\ell m}$.

to (c) : you may use $L_x = \frac{(L_+ + L_-)}{2i}$, where L_+ and L_- are the raising and lowering operators, resepectively.

(2) <u>Time-independent Perturbation</u>

The spin Hamiltonian of a spin - 1/2 particle is given by

$$\hat{H} = \mu_s \vec{S}.\vec{B}$$

Suppose, a particle is placed in an external magnetic field

$$\vec{B} = (0, B_2, B_o),$$

where $B_o >> B_2$.

(a) First, determine the energy eigenvalues of $\vec{H}$ exactly.

(b) Secondly, apply perturbation theory to calculate the energy eigenvalues through second order in (B_2/B_o), and compare the results with the exact values obtained in part (a). Do they agree ?

Hint: in part (a), during your trip, define $\omega_o = - \mu_s B_o$ and $\omega_2 = \mu_s B_2$

(3) <u>Variational Method</u>

The Hamiltonian of a one-dimensional oscillator of mass m is given by

$$\hat{H} = \frac{\hat{P}_x^2}{2m} + b\hat{x}^4, \quad (b > 0)$$

The exact value of the ground-state energy is

$$E_o = 1.06\, b^{1/3} \left(\frac{h^2}{2m}\right)^{2/3}$$

Apply the variational method to estimate the ground-state energy of the oscillator, and to see how close this result will be to the exact one.

<u>Hint</u>: You need to work with the formalism in chapter 7, in particular :

$\hat{a}^+ |n> = \sqrt{n+1} \quad |n+1>$

$\hat{a}^+ |n> = \sqrt{n} \quad |n-1>$

$<n' |a^+ |n> = \sqrt{n+1} \quad \delta_{n'}, n+1$

$<n' |a |n> = \sqrt{n} \quad \delta_{n'}, n+1$

$a |0> = 0$

Express $\hat{x}$ and $\hat{P}_x$ in terms of $\hat{a}^+$ and $\hat{a}$.

(4) Scattering Theory

(a) For a short-range potential, verify that outside the range of the potential, the wave function

$$U(r,\theta) = \frac{1}{r}\left(1 + \frac{i}{kr}\right) e^{ikr} \cos\theta$$

represents an outgoing P-wave ($\ell = 1$).

(b) Consider a beam of particles represented by a plane wave e^{ikz}, scattered by an impenetrable sphere of radius a, and assume $k a << 1$.
Consider only S and P components in the scattered wave (d-waves excluded).

Show that to the order of $(ka)^2$, the differential cross section at an angle θ is:

$$\frac{d\sigma}{d\Omega} = a^2 \left\{ 1 - \frac{1}{3} (ka)^2 + 2\,(ka)^2 \cos\theta \right\}$$

Hints:

In part (a) : the term "outside" means $V = 0$. Ask yourself: which equation should be satisfied by the radial part of the wave function.
Note that $P_1(\cos\theta) = \cos\theta$

In part (b): somewhat tricky.
First obtain a general expression of the scattering amplitude which will contain two constants.
Study the behavior of the wave function at $r \to a$ to determine the constants.

Note that excluding the d-waves contribution means that $\cos^2\theta = 1/3$ (average over all direction).