Problem 1: A system with two degrees of freedom x and y has the Lagrangian

$$L = x\dot{y} + y\dot{x}^2 + \dot{y}\dot{x}$$

Derive Lagrange's equations. Obtain the Hamiltonian $H(x, y, p_x, p_y)$. Derive Hamilton's equations and show that they are equivalent to Lagrange's equations.

Problem 2: Write down the properties of Poisson brackets. Let $f(p, q, t)$ be some function of co-ordinates, momenta and time, express its time evolution using Poisson brackets. Prove that if $f\ (p, q, t)$and $g(p, q, t)$ are two integrals of motion, then their Poisson bracket is likewise an integral of motion. Determine the Poisson brackets formed by the components of angular momenta.

Problem 3:

Is the following transformation canonical (verify your answer)?

$$P = \frac{1}{2}\left(p^2 + q^2\right), \quad Q = \tan^{-1}\left(\frac{q}{p}\right)$$

Problem 4: The angular velocity is expressed using the Euler angles by

$$\boldsymbol{\omega} = \begin{pmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{pmatrix}$$

Express the kinetic energy T of a rotating rigid body, using the principal moments of inertia I_1, I_2, I_3, in terms of the Euler angles. Assuming there is no torque, write down Lagrange's equation with respect to ψ to obtain the third component of the Euler's equation of motion.

Problem 5: Write down the action for an electromagnetic field A_μ with sources. Using the principle of least action together with the Bianchi identity, derive Maxwells equations of electromagnetism in terms of the electric and magnetic fields. Write down the Lagrangian and derive the relativistic equation of motion for a charge particle in an electromagnetic field. Show that this relativistic equation reproduces the Lorentz force law. Working in cylindrical polar coordinates, consider the vector potential $\mathbf{A} = (0, rg(z), 0)$ where $g(z) > 0$. Obtain two constants of motion. Show that if the electron is projected from a point (r_0, θ_0, z_0) with velocity $\dot{r} = \dot{z} = 0$ and $\dot{\theta} = 2eg(z_0)/m$, then, provided some condition is satisfied, it will describe a circular orbit.