AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH 251
FINAL EXAM
FALL 2007-2008
Closed Book, 2 HOURS

WRITE YOUR ANSWERS ON THE QUESTION SHEET

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	**30**	
2	**15**	
3	**15**	
4	**40**	
TOTAL	**100**	

1. (30 points) Consider the following 5 points:

x	1	2	3	4	5
f(x)	1	4	11	17	23

Find the cubic spline of each subinterval in the following way:

(a) Find the values of the moments (w_i) by solving the system $Aw = r$

- Specify the elements of matrix A and vector r:

- Apply Naive Gauss Elimination to get an upper triangular system.

(Continued...)

- Perform backward substitution to get the solution of the system $Aw = r$.

(b) Find the derivatives (z_i)

- Find the value of z_0.

- Using z_0 compute the remaining values of z.

2. (15 points) Let

$$a = (1, a_{q-1}, a_{q-2}, \cdots\cdots, a_1, a_0), a_i \in \{0,1\}$$
$$b = (1, b_{r-1}, b_{r-2}, \cdots\cdots, b_1, b_0), b_i \in \{0,1\}$$

be the binary representations of 2 base 10 numbers M and N respectively. The base 10 numbers M and N corresponding to the bits given by a and b are:

$$M = 2^q + a_{q-1}2^{q-1} + \cdots\cdots + a_1 2 + a_0$$
$$N = 2^r + b_{r-1}2^{r-1} + \cdots\cdots + b_1 2 + b_0$$

such that $N \leq M \Leftrightarrow r \leq q$.

$$\begin{aligned} M * N = & \; (2^r + b_{r-1}2^{r-1} + \cdots\cdots + b_1 2 + b_0) * N \\ = & \; b_0 N \qquad + \\ = & \; b_1 2N \qquad + \\ & \; \vdots \\ = & \; b_{r-1}2^{r-1}N + \\ = & \; 2^r N \end{aligned}$$

(a) Complete the following `MATLAB` program which evalutes $M * N$ using nested multiplication:

```
function [P]= MbyN(r,b,N)
T = N;
P = 1;
%%% Using the nonzero elements of vector b compute M*N
for i=







end
```

(b) Prove that $r \approx \log_2 N$

3. (15 points) Consider the problem of finding $r = \frac{1}{\sqrt{R}}$.

(a) Give the function $f(x)$ for which r is the unique positive solution, such that the computation of $f(x)$ does not require any division.

(b) Write Newton's iteration formula that gives the sequence $\{x_n\}$ that would converge to r. Give a graphic justification into why the the sequence $\{x_n\}$ converges for every $x_0 > 0$.

(c) Prove that:

$$x_{n+1} - r = \frac{(x_n - r)^2}{2x_n}$$

4. The function values of $f(x)$ are arranged in a table as follows:

i	x_i	$f(x_i)$
0	0.000	1.0000000
1	0.125	1.1108220
2	0.250	1.1979232
3	0.375	1.2663800
4	0.500	1.3196170
5	0.625	1.3600599
6	0.750	1.3895079
7	0.875	1.4093565
8	1.000	1.4207355

(a) Give the formula for the composite trapezoid rule, $T(h)$ to approximate the integral $I = \int_0^1 f(x)dx$ and write the expression of $I - T(h)$ in terms of powers of h.

(b) Derive subsequent, Romberg integration formulae, $R^1(h)$, $R^2(h)$, $R^3(h)$.

(c) Based on the above $f(x_i)$ data, approximate the integral $I = \int_0^1 f(x)dx$, by filling the following table:

h	$T(h)$	$R^{(1)}(h)$	$R^{(2)}(h)$	$R^{(3)}(h)$
$h_0 = 1$				
$h_0/2 = 0.5$				
$\frac{h_0}{4} = 0.25$				
$\frac{h_0}{8} = 0.125$				

(d) Using the forward Difference formula to approximate $f^{'}(0)$, followed by Richardson extrapolation, find the best approximation to $f^{'}(0)$ starting with $h_0 = 0.5$. For that purpose, derive the formulae of the following table and fill its entries:

h	$\phi(h)$	$\phi^{(1)}(h)$	$\phi^{(2)}(h)$
$h_0 = 0.5$			