AMERICAN UNIVERSITY OF BEIRUT
Faculty of Arts and Sciences
Mathematics Department

MATH 251
FINAL EXAM
FALL 2007-2008
Closed Book, 2 HOURS

WRITE YOUR ANSWERS ON THE QUESTION SHEET

STUDENT NAME	
ID NUMBER	

Problem	Out of	Grade
1	**20**	
2	**10**	
3	**20**	
4	**15**	
5	**25**	
6	**10**	
TOTAL	**100**	

The next 2 questions deal with the **natural cubic spline** $S(x)$ on the following table of 5 data points for a function $f(x)$:

i	0	1	2	3	4
x_i	1	2	3	4	5
$f(x_i)$	1	4	11	17	23

1. (20 points) Let $w_i = S''(x_i)$ be the values of the moments of the cubic spline.

   (a) (10 points) Specify the system $Aw = r$.

(b) (10 points) Solve $Aw = r$ using Naive Gauss Elimination followed by backward substitution.

2. (10 points) On the basis of w, **give the formulae only** of:

   (a) (5 points) The spline derivatives: z_0 followed by $z_i = S'(x_i), i = 1 \cdots 4$.

   (b) (5 points) The expression of $S_i(x)$, the spline $S(x)$ restricted for $x \in [x_i, x_{i+1}]$.

3. (20 points) Consider the problem of finding $r = \frac{1}{\sqrt{R}}$.

   (a) (5 points) Give the function $f(x)$ for which r is the unique positive solution, such that the <u>computation of $f(x)$ does not require any division.</u>

   (b) (7 points) Write Newton's iteration formula that gives the sequence $\{x_n\}$ that would converge to r. Give a graphic justification into why the the sequence $\{x_n\}$ converges for every $x_0 > 0$.

(c) (8 points) Prove that:

$$x_{n+1} - r = \frac{(x_n - r)^2}{2x_n}$$

4. (15 points) Based on a set of data:

$$\{(x_i, y_i = f(x_i))|i = 0, 1, ...n\}$$

where $nh = 1$ and $x_i = ih$, $\forall i$. Consider the composite trapezoid rule, $T(h)$ to approximate the integral $I = \int_0^1 f(x)dx$.

(a) (5 points) Give the formula for $T(h)$ and the expression of $I - T(h)$ in terms of powers of h.

(b) (5 points) Prove the following formula:

$$T(h/2) = (T(h) + h\Sigma_{i=0}^{n-1} f(x_i + h/2))/2 \quad (1)$$

(c) (5 points) Complete the following `MATLAB` program which takes as input two vectors x and y and approximates $I = \int_0^1 f(x)dx$ using the composite trapezoidal rule.

```
function T = trapezoid(x,y)
%  x = (x(1), x(2),........, x(n))
%  y = (y(1), y(2),........, y(n))
```

The next 2 questions deal with the following table of values for a function $f(x)$, arranged in a table as follows:

i	x_i	$f(x_i)$
0	0.000	1.0000000
1	0.125	1.1108220
2	0.250	1.1979232
3	0.375	1.2663800
4	0.500	1.3196170
5	0.625	1.3600599
6	0.750	1.3895079
7	0.875	1.4093565
8	1.000	1.4207355

5. (25 points) We intend to approximate $I = \int_0^1 f(x)dx$, using Romberg integration based on the composite trapezoid rule $T(h)$.

   (a) (10 points) Derive subsequent, Romberg integration formulae, $R^1(h)$, $R^2(h)$, $R^3(h)$ by giving expressions of the errors: $I - R^1(h)$, $I - R^2(h)$, $I - R^3(h)$.

(b) (15 points) Based on the above $f(x_i)$ data, approximate the integral $I = \int_0^1 f(x)dx$, by filling the empty slots in the following table:
(You may use formula (1) of 4.b, to fill the column of $T(h)$.)

h	$T(h)$	$R^1(h)$	$R^2(h)$	$R^3(h)$
$h_0 = 1$		$\times$	$\times$	$\times$
$h_0/2 = 0.5$			$\times$	$\times$
$\frac{h_0}{4} = 0.25$				$\times$
$\frac{h_0}{8} = 0.125$				

6. (10 points) Using the forward Difference formula to approximate $f^{'}(0)$, followed by Richardson extrapolation, find the best approximation to $f^{'}(0)$ starting with $h_0 = 0.5$. For that purpose, derive the formulae of the following table and fill the empty slots in its entries:

h	$\phi(h)$	$\phi^{(1)}(h)$	$\phi^{(2)}(h)$
$h_0 = 0.5$		$\times$	$\times$
			$\times$