MATH 102 Exam 1, Oct 16, 2018

1) a) 3P

$$f'(x) = \frac{d}{dx}\left(x^2 + \ln(\ln(x))\right)$$
$$= 2x + \frac{1}{\ln(x)} \cdot \frac{1}{x} = 2x + \frac{1}{\ln(x^x)}$$

b) 2P $\ln(x) > 0$ for $x > 1$, hence $f'(x) > 0$ for $x \in (1, \infty)$.

3P A <u>strictly</u> increasing function is one-to-one.

c) $\lim_{x \to 0^+} \ln(x) = -\infty$, $\ln(1) = 0$ and $\lim_{x \to \infty} \ln(x) = \infty$ are known.

3P

$$\lim_{x \to 1^+} f(x) = 1^2 + \lim_{x \to 1^+} \ln(\ln(x))$$
$$= 1 + \lim_{y \to 0^+} \ln(y) = -\infty$$

2P

$$\lim_{x \to \infty} f(x) = \infty + \infty = \infty$$

2P By the <u>Intermediate Value Theorem</u>, f assumes all values in $(-\infty, \infty)$.

d) f is one-to-one and differentiable:

$$\underbrace{(f^{-1})'(e^2) = \frac{1}{f'(f^{-1}(e^2))}}_{2P} \underbrace{= \frac{1}{f'(e)} = 2e + \frac{1}{e}}_{3P}$$

2)

$$\frac{df}{dx} = f \cdot \left(x + 2 \cdot \ln\left(x^x\right)\right)$$

separation of variables $\Longrightarrow$

4P $$\int \frac{df}{f} = \int \left(x + 2\ln(x) \cdot x\right) dx$$

2P l.h.s. $\ln(f)$

r.h.s. $\frac{1}{2}x^2 + \int \underbrace{\ln(x) \cdot 2x \; dx}_{}$ int. by parts

$$= \ln(x) \cdot x^2 - \int \frac{1}{x} \cdot x^2 \, dx$$

4P $$= \ln(x) \cdot x^2 - \frac{1}{2}x^2 + C$$

$$= \ln(x) \cdot x^2 + C \quad , \qquad C \in \mathbb{R} \text{ const.}$$

2P $\xrightarrow{\exp}$ $f(x) = \exp\left(\ln(x) \cdot x^2\right) \cdot e^C$

$f(1) = \exp\left(\ln(1) \cdot 1^2\right) \cdot e^C = e^C \overset{!}{=} 1$

$\Rightarrow$ choose $C = 0$.

2P $f(x) = \exp\left(\ln(x) \cdot x^2\right) = x^{x^2}$

3)

$$y(x) = \frac{(2x+1)\cdot \exp(5x)}{(x+2)\cdot(x-3)\cdot(x+4)}$$

2P $\quad \frac{d}{dx} \ln(y(x)) = \frac{y'(x)}{y(x)}$

2P $\quad = \frac{d}{dx}\left(\ln(2x+1) + 5x - \ln(x+2) - \ln(x-3) - \ln(x+4)\right)$

2P $\quad = \frac{1}{2x+1}\cdot 2 + 5 - \frac{1}{x+2} - \frac{1}{x-3} - \frac{1}{x+4}$

2P $\quad \longrightarrow \quad y'(x) = \frac{(2x+1)\cdot \exp(5x)}{(x+2)\cdot(x-3)\cdot(x+4)}\cdot\left(\frac{2}{2x+1} + 5 - \frac{1}{x+2} - \frac{1}{x-3} - \frac{1}{x+4}\right)$

4) a) apply Bernoulli – de l'Hôpital

$$\lim_{h \to 0} \frac{f(x+h) - 2 \cdot f(x) + f(x-h)}{h^2}$$

$$\underset{\frac{0}{0}}{=} \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$ 2P

$$\underset{\frac{0}{0}}{=} \lim_{h \to 0} \frac{f''(x+h) + f''(x-h)}{2} = f''(x)$$ 3P

b) $$\lim_{x \to \infty} x^{x^{1/x}} = \lim_{x \to \infty} \exp\left(\ln(x) \cdot x^{1/x}\right)$$

$$= \lim_{x \to \infty} \exp\left(\ln(x) \cdot \exp\left(\ln(x) \cdot \frac{1}{x}\right)\right)$$ 2P

since $\lim\limits_{x \to \infty} \frac{\ln(x)}{x} \underset{\frac{\infty}{\infty}}{=} \lim\limits_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$ we have 2P

$\lim\limits_{x \to \infty} \exp\left(\ln(x) \cdot \frac{1}{x}\right) = 1$. Hence

$$\lim_{x \to \infty} \ln(x) \cdot \exp\left(\ln(x) \cdot \frac{1}{x}\right) = \infty \Rightarrow$$ 3P

$$\lim_{x \to \infty} x^{x^{1/x}} = \infty$$

5)

$$\frac{x^2}{x^2-x-6} = \frac{x^2-x-6+x+6}{x^2-x-6}$$

$$= 1 + \frac{x+6}{x^2-x-6}$$

$$= 1 + \frac{x+6}{(x-3)\cdot(x+2)}$$

2P

partial fractions expansion with Heaviside method:

$$= 1 + \frac{A}{x-3} + \frac{B}{x+2}$$

2P

$$A = \frac{3+6}{3+2} = \frac{9}{5}$$

2P

$$B = \frac{-2+6}{-2-3} = -\frac{4}{5}$$

2P

$$\int \frac{x^2}{x^2-x-6}\,dx = x + \frac{9}{5}\ln|x-3| - \frac{4}{5}\ln|x+2| + \text{const.}$$

2P

2P 6) $$\int \sin^8(x)\cos^3(x)\,dx = \int \sin^8(x)\cdot(1-\sin^2(x))\cdot\cos(x)\,dx$$

2P $$\underset{u=\sin(x),\ du=\cos(x)\,dx}{=} \int u^8(1-u^2)\,du = \frac{u^9}{9} - \frac{u^{11}}{11} + \text{const.}$$

2P $$= \frac{\sin^9(x)}{9} - \frac{\sin^{11}(x)}{11} + \text{const.}$$

7)

$$\int e^{x/2} \cdot \sqrt{1-e^x}\, dx$$

4P

$$\underset{\substack{y=e^{x/2} \\ dy = \frac{1}{2}e^{x/2}dx}}{=} 2\int \sqrt{1-y^2}\, dy \underset{\substack{y=\sin(z) \\ dy = \cos(z)\,dz}}{=} 2\int \sqrt{1-\sin^2(z)}\, \cos(z)\, dz$$

where $z \in [0, \pi/2]$

(or directly substitute $\sin(z) = e^{x/2}$)

2P

$$= 2 \cdot \int \cos^2(z)\, dz = \int (1+\cos(2z))\, dz$$

2P

$$= z + \frac{\sin(2z)}{2} + C = z + \sin(z)\cdot\cos(z) + C$$

2P

$$= \sin^{-1}(y) + y \cdot \sqrt{1-y^2} + C$$

$$= \sin^{-1}(e^{x/2}) + e^{x/2}\sqrt{1-e^x} + C$$

8) a)

3P

$$\int 1 \cdot f^{-1}(y)\,dy = y \cdot f^{-1}(y) - \int y \cdot \frac{1}{f'(f^{-1}(y))}\,dy$$

$$\uparrow \qquad \downarrow$$

$$= y \cdot f^{-1}(y) - \int f(x)\,dx$$

$$x = f^{-1}(y)$$

$$dx = \frac{1}{f'(x)}\,dy$$

b)

3P

$$= y \cdot f^{-1}(y) - \int (x^2 - 2x)\,dx$$

$$= y \cdot f^{-1}(y) - \frac{x^3}{3} + x^2 + C$$

$$= y \cdot f^{-1}(y) - \frac{(f^{-1}(y))^3}{3} + (f^{-1}(y))^2 + C$$

3P

Determine $f^{-1}(y)$: $x^2 - 2x = y$

$$\Longleftrightarrow (x-1)^2 - 1 = y$$

$$\Longleftrightarrow x = \pm\sqrt{y+1} + 1$$

Since $x > 2$, $x = \sqrt{y+1} + 1$.

1P

$$\Rightarrow \int f^{-1}(y)\,dy$$

$$= y \cdot (\sqrt{y+1} + 1) - \frac{(\sqrt{y+1}+1)^3}{3} + (\sqrt{y+1}+1)^2 + \text{const}$$

$$= y + y\sqrt{y+1} - \frac{(\sqrt{y+1})^3}{3} - \cancel{(y+1)} - \sqrt{y+1}$$

$$+ \cancel{y+1} + 2\sqrt{y+1} + \text{const.}$$

$$= y + \frac{2}{3}(y+1)^{3/2} + \text{const.}$$

g) 2P

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{e^{-x}}{\sqrt{x}}\,dx + \lim_{b\to\infty} \int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$$

comparison principle for $0 < a \leq x \leq 1$:

2P

$$\frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$$

2P

$$\int_a^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_{x=a}^{1}$$

$$= 2 - 2\sqrt{a} \xrightarrow[(a\to 0)]{} 2 \text{ exists.}$$

comparison principle for $1 \leq x \leq b < \infty$

2P

$$\frac{e^{-x}}{\sqrt{x}} \leq e^{-x}$$

2P

$$\int_1^b e^{-x}\,dx = -e^{-x}\Big|_{x=1}^{b}$$

$$= e^{-1} - e^{-b} \xrightarrow[(b\to\infty)]{} \frac{1}{e} \text{ exists}$$

$\Longrightarrow$ The integral $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ exists.