1. **(8 points) Find the limit of the following sequence** $a_n = (e^n - 1)^{\frac{1}{n}}$.

$\ln a_n = \ln (e^n - 1)^{1/n} = \frac{1}{n} \ln(e^n - 1)$

$= \frac{\ln(e^n - 1)}{n} = \lim \frac{\frac{e^n}{e^n - 1}}{1} = 1$

$\Rightarrow a_n \to e$

2. Determine if the following series converge or diverge. **Justify your answers**

a. (8 points) $\sum_{n=2}^{+\infty} \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}} \Rightarrow$ cv p-series with $p > 1$

b. (8 points) $\sum_{n=1}^{+\infty} \left(1 - \frac{2}{n}\right)^{3n} \longrightarrow e^{-6} \neq 0 \Rightarrow$ div

by n^{th} term test

c. (8 points) $\sum_{n=1}^{+\infty} (-1)^n \frac{\sin n}{n^2}$ not AST because $\sin n$ is not always positive

$|a_n| = \frac{\sin n}{n^2} \leq \frac{1}{n^2} \to$ cv p series $p = 2 > 1$ ($a_n \leq b_n$ abs)

d. (10 points) $\sum_{n=2}^{+\infty} \frac{1}{n (\ln n)^{3/2}}$

$\int_2^{\infty} \frac{1}{x(\ln x)^{3/2}} dx = \int \frac{du}{u^{3/2}} = \frac{-2}{\sqrt{\ln x}}$

$u = \ln x$

$du = \frac{1}{x} dx$

$= 0 + \left(\frac{1}{\sqrt{\ln 2}}\right) = 0 + 1 = 1$

$\frac{1}{n^{1.7}(\ln n)^{100}}$ multiply by $n^{1.6}$ $= \frac{1}{n^{0.1}(\ln n)^{100}} \to 0 < 1$

$\Rightarrow \frac{1}{n^{1.7}(\ln n)^{100}} < \frac{1}{n^{1.6}} \Rightarrow$ convergent

cv

$\frac{(\ln n)^x}{n^{\text{anything} > 0}} \to 0$

3. (16 points) Find the sum of the series $\sum_{n=0}^{+\infty}\left[(-1)^n\frac{(\pi)^n}{4^n}-\frac{1}{(n+1)(n+2)}\right]$

geo series

telescoping

sum of two cv series

$$\sum_{n=0}^{\infty}\left(\frac{-\pi}{4}\right)^n = \frac{1}{1+\pi/4} \quad \frac{1}{\frac{\pi+4}{4}} = \frac{4}{\pi+4}$$

4. (22 points) What is the interval of convergence of the power series $\sum_{n=0}^{+\infty}\frac{(\frac{1}{2})^n}{3n+1}(x+2)^n$.

(be sure to check convergence at the endpoints).

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{(\frac{1}{2})^{n+1}}{(3n+4)}(x+2)^{n+1}}{\frac{(\frac{1}{2})^n}{3n+1}(x+2)^n} = \left|\frac{3n+1}{3n+4}(x+2)\left(\frac{1}{2}\right)\right|$$

$$\rightarrow \left|\frac{1}{2}x+1\right| < 1$$

$$\Rightarrow -1 < \tfrac{1}{2}(x+2) < +1$$

$$-2 < x+2 < 2$$

$$-4 < x < 0$$

at $x=-4$ $\quad \sum \frac{(\frac{1}{2})^n(-2)^n}{3n+1} = \frac{(-1)^n}{3n+1}$

AST

$\therefore$ CV

at $x=0$

$$\frac{(\frac{1}{2})^n(2)^n}{3n+1} = \frac{1}{3n+1} \quad \text{direct compare}$$

$$\frac{n}{3n+1} = \frac{1}{3} > 0$$

$\Rightarrow$ div

5. The Maclaurin series for $\tan x$ is given by:

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

a. (8 points) Using the series, find the first three nonzero terms in the Maclaurin series for $f(x) = \ln(\cos x)$.
(*hint: what is* $f'(x)$?).

$f'(x) = \frac{-\sin x}{\cos x} = -\tan x$

$= x - x^3/3 - \frac{2x^5}{15} - \dots$

$f(x) = \int f'(x) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + C -$

$f(0) = 0 \Rightarrow C = 0$

b. (6 points) For what values of x can we replace $\sin x$ by $x - \frac{x^3}{6}$ with an error of magnitude no greater than 10^{-3}.

$\sin x = x - \frac{x^3}{6} + R_4(x)$

$|R_4(x)| = \frac{(x-0)^{n+1}}{n+1} \cdot f^{n+1}(c)$

$= \left|\frac{x^5}{5} \underbrace{f^5(c)}_{\leq 1}\right| < 10^{-3}$

$\Rightarrow \left|\frac{x^5}{5}\right| < 10^{-3} \cdot 5$

$\Rightarrow x^5 < 5 \times 10^{-3}$

$|x| < \sqrt[5]{5 \times 10^{-3}}$

6. (6 points) Use power series to evaluate the limit: $\lim_{x\to 0} \frac{e^{3x^2} - 1}{x^2}$.

$e^x = 1 + x + \frac{x^2}{2} + \dots$

$e^{-3x^2} = 1 - 3x^2 + \frac{9x^4}{2} + \dots$

MATHEMATICS 201

QuizI

1) Investigate the convergence or divergence of :

a) $a_n = (-1)^n \dfrac{\sin e^n}{n}$

b) $a_n = (1 + \dfrac{1}{n})^{\sqrt{n}}$

c) $\sum \dfrac{3^n + 2^n}{4^n + 5}$

d) $a_n = \sqrt{n} \sin \dfrac{1}{n^2}$

e) $\sum \dfrac{(-1)^n}{n \ln^2 n}$

2) Find S for $\sum_{k=1}^{\infty} (-1)^k \dfrac{(4)^k}{5^k} + \sum_{k=1}^{\infty} \dfrac{1}{k(k+1)}$

3)

$f(x) = \dfrac{x}{2 + 3x}$
apply taylor series for x = 1

4)

Find the radius of convergence of the following series :

$\sum \dfrac{(n!)^2}{2n!} x^n$

Solution :

1)

a) $\frac{-1}{n} < a_n < \frac{1}{n} \Rightarrow \lim a_n \to 0$ because $\sin e^n < 1$

b)

$$a_n = (1+\frac{1}{n})^{\sqrt{n}}$$

$$\sqrt{n}\ln(1+\frac{1}{n}) = \lim \frac{\ln(1+\frac{1}{n})}{\frac{1}{\sqrt{n}}} = \frac{0}{0} \quad \text{(hopital rule)}$$

$$\lim \sqrt{n}\ln(1+\frac{1}{n}) = \frac{\frac{1}{n^2}+\frac{1}{n}}{\frac{-1}{2}n^{-3/2}} = 0$$

$$\Rightarrow (1+\frac{1}{n})^{\sqrt{n}} = 1$$

c)

$$\sum \frac{3^n+2^n}{4^n+5}$$

$$\lim \frac{a_{n+1}}{a_n} = \frac{\frac{3^{n+1}+2^{n+1}}{4^{n+1}+5}}{\frac{3^n+2^n}{4^n+5}} = \frac{3}{4} < 1 \quad \Rightarrow \text{converges}$$

d)

$$a_n = \sqrt{n}\sin\frac{1}{n^2}$$

$$\lim \frac{\sqrt{n}\sin\frac{1}{n^2}}{\frac{1}{n^{3/2}}} = 1 \quad \text{but } \frac{1}{n^{3/2}} \text{ converges by integral test}$$

$$\Rightarrow \sqrt{n}\sin\frac{1}{n^2} \text{ converges}$$

e)

$\sum \frac{(-1)^n}{n\ln^2 n}$ it converges by leibneiz theorem .

it also converges absolutely by the integral test .

2)

$$\sum_{k=1}^{\infty}(-1)^k\frac{(4)^k}{5^k} + \sum_{k=1}^{\infty}\frac{1}{k(k+1)}$$

$$\sum_{k=1}^{\infty}(-1)^k\frac{(4)^k}{5^k} = \frac{-4}{5}\left(\frac{1}{1-4/5}\right) = \frac{-18}{5}$$

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} = 1 - \frac{1}{n+1} = 1$$

3)

$$\frac{x}{2+3x}$$

$$\frac{1}{3} + \frac{-2/3}{2+3x} = \frac{1}{3} - \frac{2}{3}\sum\left(\frac{3}{5}\right)^{n-1}(x-1)^{n-1}$$

4)

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 x}{(2n+1)(2n+2)} = \frac{x}{4}$$

$-4 < x < 4$

at both endpoints there is divergence

$$\sum 4^n \frac{(n!)^2}{2n!}$$

$$\lim \frac{a_{n+1}}{a_n} = \frac{4(n+1)^2}{(2n+1)(2n+2)} = \frac{2(n+1)}{(2n+1)} = \frac{2n+2}{2n+1} > 1 \quad \text{diverge}$$

MATHEMATICS 201
(1st Semester, . . -)

QUIZ 1

1) **Do the following sequences converge or diverge ?**

a) $(1-\frac{1}{n})^{\sqrt{n}}$

b) $(1+\frac{1}{n})^{n\sqrt{n}}$

c) $\dfrac{\left(\frac{15}{16}\right)^n}{\left(\frac{14}{15}\right)^n+\left(\frac{16}{17}\right)^n}$

d) $n^2(1-\cos\frac{2}{n})$

e) $(n!)^{1/n}$

2) *calculate*

a) $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

b) $\sum_{n=0} e^{-2n}$

3)

a)**Show that if** $\sum a_n$ converges $\Rightarrow$ $\sum a^2_n$ **converges .**

b)Σa_n div, $a_n > 0$ $\lim a_n \to 0$

4) **Investigate for convergence and divergence .**

a) $\sum_{n=1}\frac{1.3.5......(2n-1)}{2.4.6......2n}$

b) $\sum_{n=1}\frac{1.3.5....(2n-1)}{n(2.4.6....2n)}$

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c) $\sum_{n=1} \frac{1}{(\ln(n+1))^3}$

d) $\sum_{n=1} \frac{\ln^3 n}{n^{1.1}}$

e) $\sum_{n=1} \frac{1}{n\ln^{3/2} n(\ln\ln n)^{1/2}}$

5) **Find the domain of convergence of**

a) $\sum_{n=0} (\ln x)^n$.

b) $\sum_{n=0} 2^{nx}$

c) $\sum_{n=0} \frac{(x^2-3)}{4}\frac{(x^2-3)^{n-1}}{4}$

Solution:

1)

a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1-\frac{1}{n})^{\sqrt{n}} = (1-\frac{1}{\sqrt{n}})^{\sqrt{n}}(1+\frac{1}{\sqrt{n}})^{\sqrt{n}} = \frac{1}{e}\times e = 1$

converges .

b)

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^{n\sqrt{n}} = \lim n\sqrt{n}\times \ln\left(1+\frac{1}{n}\right) = \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n\sqrt{n}}} = \frac{0}{0}$$

undetermined, use hopital's rule

$$\Rightarrow \lim \frac{\ln(n+1) - \ln n}{\frac{1}{n^{3/2}}} = \frac{\frac{1}{n+1} - \frac{1}{n}}{\frac{-2}{3}\frac{1}{n^{5/2}}} = \frac{-3}{2}\frac{n^{5/2}}{n^2+n} = -\infty$$

$$\Rightarrow \lim\left(1+\frac{1}{n}\right)^{n\sqrt{n}} = e^{-\infty} = 0$$

c)

$$a_n = \frac{\left(\frac{15}{16}\right)^n}{\left(\frac{14}{15}\right)^n + \left(\frac{16}{17}\right)^n} = \frac{\left(\frac{15}{16}\right)^n\left(\frac{17}{16}\right)^n}{\left(\frac{14}{15}\frac{17}{16}\right)^n + 1}$$

$$\Rightarrow \lim a_n = \lim \frac{\left(\frac{15\times 17}{16\times 16}\right)^n}{\left(\frac{17\times 14}{15\times 16}\right)^n} = \lim\left(\frac{15^2}{16\times 14}\right)^n = \infty$$

d)

$$a_n = n^2(1-\cos\frac{2}{n})$$

$$\lim_{n\to\infty} a_n = \lim\frac{(1-\cos\frac{2}{n})}{1/n^2} = \frac{0}{0}$$ undetermined apply hopital rule

$$\lim_{n\to\infty} a_n = \lim\frac{\frac{-2}{n^2}\sin\frac{2}{n}}{\frac{-2n}{n^3}} = \lim\frac{\sin\frac{2}{n}}{\frac{1}{n}}$$ apply hopital rule

$$\lim\frac{\frac{-2}{n^2}\cos\frac{2}{n}}{\frac{1}{n^2}} = -2 \quad \Rightarrow \text{ converges}$$

e)

$$e^n = 1+\frac{2^2}{2!}+\frac{2^3}{3!}+\ldots+\frac{n^n}{n!}$$

$$\Rightarrow\ e^n > \frac{n^n}{n!} \quad \Rightarrow n! > \left(\frac{n}{e}\right)^n$$

$$\Rightarrow\ (n!)^{1/n} > \frac{n}{e} \quad \Rightarrow \lim (n!)^{1/n} = \lim\frac{n}{e} = \infty$$

$\Rightarrow$ diverges

2)

a)

$$s = \lim_{n\to\infty} s_n$$

$$s_n = \sum_{n=1}\frac{1}{n(n+1)} = \sum_{n=1}\frac{1}{n} - \frac{1}{n+1} = 1-\frac{1}{n+1}$$

$$s = \lim_{n\to\infty} s_n = 1$$

b)

$$\sum_{n=0} e^{-2n} = \sum\left(\frac{1}{e^2}\right)^n = \sum\left(\frac{1}{e^2}\right)\left(\frac{1}{e^2}\right)^{n-1}$$

$$s = \lim s_n = \left(\frac{1}{e^2}\right)\left(\frac{1-\left(\frac{1}{e^2}\right)^{n-1}}{1-\frac{1}{e^2}}\right) = \frac{1}{e^2-1}$$

3) Apply limit comparison test

$$\lim\frac{a^2_n}{a_n} = a_n = 0$$

$$\sum a_n \text{converge} \quad \Rightarrow \quad \lim a_n = 0$$

$$\left.\begin{array}{l}\sum a_n \text{div.} \\ a_n > 0 \\ \lim a_n \to 0\end{array}\right\} \to \sum\frac{a_n}{1+a_n} \quad \text{diverges}$$

$$\lim\frac{a_n}{(1+a_n)a_n} = 1$$

4)

a)

$$\sum_{n=1}\frac{1.3.5\ldots\ldots(2n-1)}{2.4.6\ldots\ldots 2n}$$

$$= \frac{1.3.5\ldots\ldots(2n-1)(2.4.6\ldots\ldots 2n)}{(2.4.6\ldots\ldots 2n)^2}$$

$$= \sum_{n=1}\frac{2n!}{4(n!)^2}$$

$$= \lim\frac{a_{n+1}}{a_n} = \frac{(2n+1)(2n+2)}{(n+1)^2} = 4 \qquad \text{diverges}$$

c)

$$\sum_{n=1}\frac{1.3.5\ldots\ldots(2n-1)}{n(2.4.6\ldots\ldots 2n)} > \sum_{n=1}\frac{1}{n}$$

$\sum_{n=1}\frac{1}{n}$ diverges by integral test

$$\Rightarrow \sum_{n=1}\frac{1.3.5\ldots\ldots(2n-1)}{n(2.4.6\ldots\ldots 2n)} \quad \text{diverges}$$

d)

$\sum_{n=2}\frac{\ln^3 n}{n^{1.1}}$ compare it with $b_n = n^{1.9}$

$$\lim\frac{\ln^3 n}{n^3} = 0$$

$\frac{\ln^3 n}{n^{1.1}}$ diverges

e)

$$\sum_{n=1} \frac{1}{n\ln^{3/2}(\ln\ln n)^{1/2}}$$

$n\ln^{3/2} n(\ln\ln n)^{1/2}$ is decreasing positively

$\Rightarrow \lim a_n \to 0$

integral test $\int \frac{1}{n\ln^{3/2}(n)\ln(\ln\ln n)^{1/2}} = (\ln\ln n)^{1/2}$ diverge

5)
a)

$$\sum_{n=0}(\ln x)^n = \sum_{n=0}\ln x(\ln x)^{n-1}$$

$$s = \lim\frac{\ln x(1-\ln x)}{1-\ln x}$$

converges if $-1 < \ln x < 1 \Rightarrow 1/e < x < e$

end points

for $1/e$ it diverges

for e it also diverges

b)

$$\sum_{n=0} 2^{nx} = \sum_{n=0} 2^x(2^x)^{n-1}$$

$0 < 2^x < 1$ $\qquad 2^x > 0 \ \forall x$

$-\infty \le x \le 0$

end points

for $x = 0$ $\sum 1$ converges

for $x = -\infty$ $\sum\left(\frac{1}{2^\infty}\right)^n = 0$ converges

c)

$$\sum_{n=0}\frac{(x^2-3)}{4}\frac{(x^2-3)^{n-1}}{4}$$

$$-1 < \frac{(x^2-3)}{4} < 1$$

$-\sqrt{7} \le x \le \sqrt{7}$

both endpoints converge

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MATHEMATICS 201
(1st Semester,

QUIZ 1

1) Study the convergence or the divergence of :

a) $\sum_{n=0}^{\infty}\left(\frac{n+2}{n+3}\right)^n$

b) $\sum_{n=0}^{\infty}\left(\frac{3^{2n+3}}{(3n+2)^n}\right)$

c) $\sum_{n=1}^{\infty} a_n$ *when* $a_{n+1} = \left(\frac{n+\sqrt{n}}{n+9}\right)a_n$ $\qquad a_1 = 1$

2) $f(x) = \frac{x}{3+4x}$: Find Taylor's series for f at a = -1 and $f^4(-1$

3) Find S for

a)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n! + 5^n}{2^n n!}$$

b)

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

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4) $\sum \frac{1}{n \ln n}\left(\frac{x+1}{3}\right)^n$

Find domain of convergence

a) Absolutely, conditionally

b) At x = -4

5) Find $\int_0^1 \frac{e^x - 1}{x}$ with an error of less than 0.1

Solution:

1) $a_n = \left(\frac{n+2}{n+3}\right)^n$

a) $\left(\frac{n+2}{n+3}\right)^n = \left(1-\frac{1}{n+3}\right)^n = \frac{\left(1-\frac{1}{n+3}\right)^{n+3}}{\left(1-\frac{1}{n+3}\right)^3}$

$\Rightarrow \lim_{n\to\infty} a_n = \lim\left(\frac{n+2}{n+3}\right)^n = \lim_{n\to\infty} \frac{\left(1-\frac{1}{n+3}\right)^{n+3}}{\left(1-\frac{1}{n+3}\right)^3} = \frac{1}{e} \neq 0$ diverge

b) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{3^{2n+5}}{(3n+5)^{n+1}} * \frac{(3n+2)^n}{3^{2n+3}} = \frac{\lim 9}{3n+5}\left(\frac{3n+2}{3n+5}\right)^n$

$= \lim \frac{9}{3n+5}\left(1-\frac{3}{3n+5}\right)^n = \lim \frac{9}{3n+5}\left(1-\frac{1}{n+\frac{5}{3}}\right)^n = \lim_{n\to\infty} \frac{9}{3n+5}\frac{1}{e} = 0$ converge

c) $\frac{n+\sqrt{n}}{n+9} > 1$ for $n > 81$ $\Rightarrow \lim \frac{a_{n+1}}{a_n} > 1 \Rightarrow a_{n+1} > a_n$ $\quad a_{n+1} > a_{64} \neq 0$

but $a_{81} > 0 \Rightarrow \lim a_{n+1} > a_{81} \neq 0$

Then U_n diverge

)

$$\frac{x}{3+5x}=\frac{1}{5}-\frac{3}{5}\left(\frac{1}{3+5x}\right)=\frac{1}{5}-\left(\frac{1}{5+(25/3)x}\right)=\frac{1}{5}-\left(\frac{1}{5+(25/3)(x+1)-(25/3)}\right)$$

$$=\frac{1}{5}+\left(\frac{1}{-(25/3)(x+1)+(10/3)}\right)=\frac{1}{5}+\frac{3}{10}\left(\frac{1}{1-2.5(x+1)}\right)$$

$$=\frac{1}{5}+\frac{3}{10}\sum_{n=0}^{\infty}(2.5)^n(x+1)^n=\frac{1}{5}+\frac{3}{10}+\frac{3}{10}(2.5)(x+1)+\quad\ldots$$

$f^4(-1)=??$

$a_4=\frac{f^4(-1)}{4!}\qquad\Rightarrow\quad f^4(-1)=\frac{3}{10}(2.5)^4 4!$

3)

a)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}+\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{5}{2}\right)^n=??$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}=\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{-1}{2}\right)^{n-1}=\frac{(1/2)\left(1-(-1/2)^n\right)}{1+1/2}\quad\text{(Geometric progression)}$$

$\lim_{n\to\infty} S_n = 1/3$

$$\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{5}{2}\right)^n=\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)^{n+1}}{(n+1)!}=e-1$$

$s_t=\lim_{n\to\infty}s_n e-1+(1/3)=e-(2/3)$

b)

$$\frac{1}{\sqrt{n+1}+\sqrt{n}}=\frac{1}{\sqrt{n+1}+\sqrt{n}}\times\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}=\sqrt{n+1}-\sqrt{n}\quad\text{(Telescope)}$$

$\Rightarrow\ S_n=\sqrt{n+1}-\sqrt{n}+\sqrt{n}-\sqrt{n-1}+\ \ldots\ -\sqrt{2}+\sqrt{2}-1$

$\Rightarrow\ S_n=\sqrt{n+1}-1$

$\Rightarrow\ \lim_{n\to\infty}S_n=\lim_{n\to\infty}\sqrt{n+1}-1=\infty$

$\Rightarrow$ diverges

4)

$$\sum\frac{1}{n\,\mathrm{Ln}\,n}\left(\frac{x+1}{3}\right)^n$$

$$\lim\frac{a_{n+1}}{a_n}=\frac{|x+1|}{3}$$

This series converges if and only if $\frac{|x+1|}{3}<1\ \Rightarrow\ |x+1|<3$

$\Rightarrow\ -4<x<2$

end point for $x = 2\ \Rightarrow\ \sum_{n=2}^{\infty}\frac{1}{n\ln n}$

$\frac{1}{n\ln n}>0$ integral test

$\int_2^{\infty}\frac{1}{n\ln n}dn=[\ln(\ln n)]_2^{\infty}=\infty\ \Rightarrow$ diverges

end point for $x = -4\ \Rightarrow\ \sum\frac{(-1)^n}{n\ln n}$ it conveges by leibneiz theorem.

$\Rightarrow$ it converges conditionally

b) at $x = -4$ (look at part (a))

5)

$$e^x=\sum\frac{x^n}{n!}\ \Rightarrow\ \frac{e^x-1}{x}=\frac{x}{2}+\frac{x^2}{3!}+\ldots+\frac{x^{n-1}}{n!}=\sum_{n=0}\frac{x^n}{(n+1)!}$$

$$\Rightarrow\ \int_0^1\frac{e^x-1}{x}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}=\sum_{n=1}^{\infty}\frac{x^n}{n(n!)}\quad(n_{max}=3)\quad n=3\ a_3=\frac{1}{18}<0.1$$

$$\Rightarrow\ \int_0^1\frac{e^x-1}{x}=1+(1/4)+(1/18)=\frac{47}{36}$$

MATHEMATICS 201

Time: 55 mins. (1st Semester November

QuizI

I) Investgate each of the following series for absolute convergence, conditional convergence or divergence .

$$\sum(-1)^n \cos\frac{1}{n^2} \qquad \sum\frac{(-1)^n}{(\ln n)^2} \qquad \sum\frac{n}{n^{2\tan^{-1}n}}$$

(30%)

2) Estimate $\int_0^{0.1}\frac{\cos\sqrt{x}-1}{x}dx$ with an absolute error of magnitude less than 0.0001. Is your estimate larger or smaller than the actual value ?
(15%)

3) Find the domain of convergence of $\sum\frac{(-1)^n(\cos^{-1}x)^{2n}}{4^n\ln(n+2)}$

(15%)

4) (a) Find $\lim_{n\to\infty}\left(\frac{n}{n-2}\right)^{5n}$ (if it exists) .

(b) Find $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(-1)^k 3^{4k+1}}{25^k(2k)!}$ (if it exists) .

(c) Find $f^{(99)}(0)$ if $f(x) = \sin x^3 + \cos x$.

(d) Test $\sum(n\sin(1/n)-1)$ for convergence or divergence .

(e) Find $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}$ by using the following theorem .

Theorem : $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ if this last exists . (40%)

Solution :

1)

a) $\sum(-1)^n \cos\frac{1}{n^2}$

by n^{th} term test $\lim_{n\to\infty}(-1)^n \cos(\frac{1}{n^2}) = \pm 1$ diverges

b) $\sum\frac{(-1)^n}{(\ln n)^2}$

$\frac{1}{(\ln n)^2}$ decreases $\lim_{n\to\infty}\frac{1}{(\ln n)^2} = 0$ $\frac{1}{(\ln n)^2} > 0$

it converges by Leibneiz theorem

c) $\sum\frac{n}{n^{2\tan^{-1}n}}$

$$\lim\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{(n+1)^{2\tan^{-1}n}}}{\frac{n}{n^{2\tan^{-1}n}}} = \lim_{n\to\infty}\frac{n^{2\tan^{-1}n}}{(n+1)^{2\tan^{-1}(n+1)}} = \frac{n^{2\pi/2}}{(n+1)^{2\pi/2}}$$

$= \left(\frac{n}{n+1}\right)^{\pi}$ but $(n+1) > n \Rightarrow \frac{n}{n+1} < 1$

$\Rightarrow \left(\frac{n}{n+1}\right)^{\pi} < 1$ converges

2)

$$\cos\sqrt{x} = \sum\frac{(-1)^n(\sqrt{x})^{2n}}{2n!} = \sum_{n=0}\frac{(-1)^n x^n}{2n!}$$

$$\frac{\cos\sqrt{x}-1}{x} = \sum\frac{(-1)^n x^{n-1}}{2n!}$$

$$\int_0^{0.1}\frac{\cos\sqrt{x}-1}{x}dx = \int_0^{0.1}(-1)^n\frac{x^{n-1}}{2n!} = \frac{(-1)^n x^n}{(n)(2n!)}$$

but the error is less than 0.0001

$$\Rightarrow \frac{(-1)^n(0.1)^n}{n(2n!)} = 0.0001$$

for $n = 3$ is valid $0 < n \le 3$

$$\int_0^{0.1}\frac{\cos\sqrt{x}-1}{x}dx = -\frac{0.1}{2} + \frac{0.01}{2\times 4!} - \frac{10^{-3}}{3\times 6!}$$

3)

$$\sum\frac{(-1)^n(\cos^{-1}x)^{2n}}{4^n\ln(n+2)}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 4(\cos^{-1}x)^2 < 1$$

$\Rightarrow \frac{-1}{2} < \cos^{-1}x < \frac{1}{2} \Rightarrow \frac{-\pi}{3} < x < \frac{\pi}{3}$

end points :

for $\frac{\pi}{3}$ converges by Leibneiz

for $\frac{-\pi}{3}$ converges by integral test

4)

a)

$$\lim_{n\to\infty}\left(\frac{n}{n-2}\right)^{5n} = \left(1+\frac{2}{n-2}\right)^{5n} = e^{10}$$

b)

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{(-1)^k 3^{4k+1}}{25^k(2k)!} = 3\sum\frac{(-1)^k\left(\frac{9}{5}\right)^{2k}}{2k!} = 3\cos\frac{9}{5}$$

c)

$f^{(99)}(0)$ if $f(x) = \sin x^3 + \cos x$

$$\sin x^3 + \cos x = 1 - \frac{x^2}{2!} + x^3 + \frac{x^4}{4!} + \ldots = \sum(-1)^n\left(\frac{x^{4n+1}}{(2n+1)!} + \frac{1}{2!}\right)$$

$$f^{99}(0) = \underset{\sin x^3}{f^{99}(0)} + \underset{\cos x}{f^{99}(0)} = \frac{(-1)^{99}}{199!}(99!) + \frac{(-1)^{99}}{198!}(99!) = \frac{-99!}{199!}(200)$$

d)

$\sum(n\sin(1/n)-1)$

$\sin\frac{1}{n}\le\frac{1}{n} \Rightarrow n\sin\frac{1}{n}\le n\frac{1}{n}=1$ $\qquad n\sin\frac{1}{n}-1\le 0$

$\sum n\sin\frac{1}{n}-1\le\sum 0$ converge by integal test

e)

$E=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=??$

$\lim\sqrt[n]{n!}=\lim\frac{(n+1)!}{n!}=n+1$

$\Rightarrow E=\lim_{n\to\infty}\frac{n+1}{n}=1$

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Time : 50 Mins. | Mathematics 201 | Name :
ID # :
Section :

Quiz I

Problem 1. 10 pts.

Investigate each of the following series for convergence or divergence

a. $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{2n}$

b. $\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^{5/4}}$

Problem 2. 8 pts.

Use the binomial theorem to estimate $\sqrt{1.2}$ with an error of magnitude less than 0.001 .

Problem 3. 8 pts.

Suppose that $a_n > 0$, and $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} \frac{a_n}{a_n^2 + 1}$ converges.

(Hint : Use the comparison or limit comparison test)

Problem 4. 8 pts.

Find the interval of convergence for the following power series

$$\sum_{n=1}^{\infty} \frac{n!}{1.4.7....(3n-2)} x^n$$

Circle the correct answer in each of the following problems (Problem 5 to Problem 8) . [4 points for each correct answer, -1 for each wrong answer, and 0 for no answer].

Problem 5. 4 pts.

The sequence $a_n = \dfrac{(\ln n)^5}{n^{1/n}}$

a. Converges to 120

b. Converges to 0

c. Diverges

d. Converges to 5

Problem 6. 4 pts.

The series $\sum_{n=1}^{\infty} \dfrac{(\cos n\pi)n^{2/5}}{n^{3/2}}$

a. Converges absolutely

b. Converges conditionally

c. Diverges

Problem 7. 4 pts.

The sum of the series $\sum_{n=2}^{\infty} \dfrac{(-1)^n \pi^{2n}}{(2n)!}$

a. π^2

b. $-2 + (\pi^2 / 2)$

c. -1

d. None of the above

Problem 8. 4 pts.

The interval of convergence for the power series $\sum_{n=2}^{\infty} \dfrac{\ln n}{n} x^n$ is

a. $-1 \le x \le 1$

b. $-1 \le x < 1$

c. $-1 < x \le 1$

d. $-\infty < x < \infty$

Hamdan

Multi Dimensional Calculus

Exam I

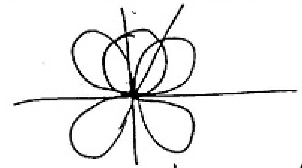
Fall 1999

Nov 2, 1999

Name: Key :

1. Consider the two polar curves: $r = \sin 2\theta$ and $r = \sin\theta$

(a) Find all points of intersection of the 2 curves



intersection : Origin $(0, \text{any } \theta)$

$r = r$

$\sin 2\theta = \sin\theta$

$2\theta = \pi - \theta \Rightarrow 3\theta = \pi$

$\theta = \pi/3$ or

$\theta = 2\pi/3$ (sym.)

$\Rightarrow$ points are $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right), \left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$

(b) *Just write* the integrals representing the area of the region inside the curve $r = \sin\theta$ and outside the curve $r = \sin 2\theta$. Do not evaluate the integral.

$$2\int_{\pi/3}^{\pi/2} \frac{1}{2}\left[(\sin\theta)^2 - (\sin 2\theta)^2\right] d\theta$$

2. Sketch the polar curve: $r\sin^2\theta = \cos\theta + \frac{3}{r}$

Multiply by r :

$r^2\sin^2\theta = r\cos\theta + 3$

$y^2 = x + 3$: parabola

$x = y^2 - 3$