Physics Dept. | Physics 302 Final Exam | June 24, 1997 Time: 3 hours

Name:____

I.D. No.:________________________

All problems in this exam are obligatory

Problem	Grade
1. Classical ideal gas	....
2. Maxwell-Boltzman n statistics	....
3. Fermi gas and spin paramagnetic.	....
4. Bose gas	....
5. Debye Model	....
6. Simple Ising model	....
Total:	

1. Consider an ideal classical gas of N particles in thermal equilibrium at temperature T in a containter of volume V. The gas is subjected to a uniform gravitational field where the gravitational acceleration is in the negative z-direction.

   a) Write down the single-particle energy assuming the extension of the volume to be small, such that the potential of a particle is constant.

   b) Calculate the chemical potential μ of an element of this gas as a function of (T,P,z), where z denotes the height, and P the pressure.

   c) With the requirement that μ is independent of z, obtain the atmospheric pressure P = P (T, z).

   No internal degrees of freedom are considered in this problem, only translation.

Hints: your can use the eq. of state : $P = \frac{N}{V} k_B T$

Stirling formula is useful: $\ell n\, N! \approx N \ell N\, N - N$

Useful: $\int_0^{-\infty} x^2 e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{4a^3}$ for a > 0.

2. Consider N non-interacting particles obeying the Maxwell-Boltzman statistics. They may exist in one of three non-degenerate energy levels of energies -E, 0, +E. The system is in contact with a heat reservoir at temperature T.

   (a) What is the maximum entropy of the system?
   (b) What is the minimum possible energy of the system?
   (c) What is the partition function of the system?
   (d) What is the most probable energy of the system?
   (e) Find $\int_0^{\infty} \frac{C(T)}{T} dT$, where C (T) is the heat capacity of the system.

3. An electron in a external magnetic field H has an energy $\pm \mu_B H$ depending on whether the spin magnetic moment is parallel or anti-parallel to the field. For a system of free electrons assumed to be completely degenerate (zero temperature limit) calculate:

   (a) The total magnetic moment M in terms of μ_o (Fermi-Energy) and H.
   (b) Consider the case $\mu_o >> \mu_B H$ and obtain an approximate for M.
   (c) Obtain the spin paramagnetic susceptibility χ, and show that $\chi = \frac{3}{2} \mu_B^2 \frac{N}{V} \frac{1}{\mu_o}$

4. Consider a system of massless bosons with single-particle energies $\varepsilon(\vec{K}) = \hbar C K$.

   (a) Find the grand canonical potential $\Omega(T,V,\zeta)$ of this system ($\zeta \equiv e^{\beta\mu}$)
   (b) Determine the pressure P, particle number density n and internal energy u as a function of (T, V, ζ)
   (c) Show that as $N \to \infty$, $V \to \infty$, $n = \frac{N}{V}$ finite, $u \to 3PV$
   (d) Determine the critical temperature T_C and critical particle number density n_C of the Bose-Einstein condensation.
   (e) Determine the transition curve $P_C = F(n_C)$ in the $p - \frac{1}{n}$ plane.

5. Consider the normal vibrations of atoms in a solid, that is the atoms are assumed to vibrate about their equilibrium positions with small amplitudes. In the Debye model, the frequencies of the uncoupled oscillators are distributed according to the density of state:

$$D(\omega) = \begin{cases} \alpha\,\omega^2 \; for \;\; \omega \langle \omega_D \\ 0 \;\; otherwise \end{cases}$$

where $\alpha \equiv \frac{9N}{\omega_D^3}$ and ω_D is Debye frequency. $D(\omega)$ is the number of vibrational modes having frequencies between ω and $\omega + d\omega$, and ω_D is fixed by

$$\int_0^\infty D(\omega)\, d\omega = 3N \quad \text{(N is the number of atoms in the lattice)}$$

   (a) Calculate the specific heat at constant volume C_V.
   (b) Discuss the behavior of C_V at high temperatures $(\hbar\omega_D / K_B T) << 1$ and low temperatures $(\hbar\omega_D / K_B T >> 1)$.

   Useful to know : $\int_0^\infty \frac{x^4\, dx}{(e^x - 1)^2} = \frac{4\pi^4}{15}$.

6. A system consists of three spins in a chain (simple Ising system) each having spin $S = \frac{1}{2}$ and is coupled by the nearest neighbors interactions. Each spin has a magnetic moment $\vec{\mu} = 2\mu\vec{s}$. The system is placed in an external magnetic field $\vec{H} = (0, 0, H)$, and is in thermal equilibrium at temperature T. The Hamiltonian is approximated by that of the Ising model :

$$H = J \sum_{ij} s_i s_j - 2\mu H \sum_i s_i$$

(J and μ positive)

(a) Make a list of the possible energies of the system.
(b) How many energy levels does the system have when $H = 0$.
(c) Are some of the energy levels degenerate? If yes what is the degree of degeneracy?
(d) Find the partition function $Z(T, H)$
(e) Find the magnetization $M(T, H)$. What is the approximate expression of M in case of $K_B T >> \mu H$?