Physics 309J, Final Exam, January 22 2008
Time: 3 hours

1. A frame F' with coordinates (x', t') is moving with a velocity v with respect to frame F with coordinates (x, t)

$$x' = Ax + B(ct)$$
$$ct' = Cx + D(ct)$$

such that the distance $x^2 - (ct)^2 = x'^2 - (ct')^2$ is invariant. Find the conditions on A, B, C, and D. Using the fact that the when the origins of the two frames coincide at $x' = 0$ then $x = vt$, express A, B, C, and D in terms of v.

2. Define the conformal tensor by

$$C_{\mu\nu\kappa\lambda} = R_{\mu\nu\kappa\lambda} + a\left(g_{\mu\kappa}R_{\nu\lambda} - g_{\nu\kappa}R_{\mu\lambda} - g_{\mu\lambda}R_{\nu\kappa} + g_{\nu\lambda}R_{\mu\kappa}\right) + b\left(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa}\right)R$$

such that all contractions of this tensor vanish, and in particular $g^{\mu\kappa}C_{\mu\nu\kappa\lambda} = 0$ and $g^{\nu\lambda}C_{\mu\nu\kappa\lambda} = 0$. Find a and b.

3. Find the Christoffel symbols, Riemann curvature tensor and Ricci tensor components for the space-time with metric

$$ds^2 = -e^{2A(r)}dt^2 + dr^2$$

Determine the function $A(r)$ so that the equations $R_{\mu\nu} = 0$ are satisfied.

4. In flat Minkowski space time find the ten Killing vectors corresponding to invariance of the metric under translations and rotations. Express these in terms of the basis vectors $\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

5. For a $k = -1$ Friedman cosmology with $p = \rho = 0$ show that the line element becomes

$$ds^2 = -dt^2 + t^2\left[d\chi^2 + \sinh^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

Exhibit an explicit coordinate transformation to show that this metric describes a Minkowski space.

6. Solve the first order Friedman equation

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2}$$

for $R(t)$ when matter is dominated by radiation for the three possibilities $k = 1, 0, -1$.

7. A particle is falling radially from rest at $r = a$ in Schwarzschild geometry. Derive the equations of motion relating t, r and τ. Hint: Use the first integrals of energy and angular momentum conservation, and use the change of variable $r = \frac{a}{2}(1 + \cos\eta)$ to express dt in terms of $d\eta$ but do not do the t integral.