Formulae: $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ **(all real *x*)**

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ **(all real *x*)**

1) (10 points) Use series to evaluate (DO NOT USE L'HOPITAL'S RULE).

a) $\lim_{x \to 0} \frac{e^{2x} - (1+2x)}{x^2}$

$e^{2x} = 1 + 2x + 2x^2 + \cdots + \frac{x^{2n}}{(2n)!}$

$\Rightarrow \lim_{x \to 0} \frac{1 + 2x + 2x^2 + \frac{8x^3}{6} + \cdots - (1+2x)}{x^2} = \lim_{x \to 0} \frac{2x^2 + \frac{8}{6}x^3}{x^2}$

10

$\Rightarrow \lim_{x \to 0} \frac{\cancel{x^2}\left(2 + \frac{8x}{6} + \cdots\right)}{\cancel{x^2}} = 2.$

b) $\lim_{x \to \infty} (x+1) \sin \frac{1}{x+1}$

$\sin \frac{1}{x+1} = \frac{1}{x+1} - \frac{1}{(x+1)^3 \times 3!} + \frac{1}{(x+1)^5 \times 5!} + \cdots$

$\Rightarrow \lim_{\infty} (x+1) \left(\frac{1}{x+1} - \frac{1}{(x+1)^3 \times 3!} + \frac{1}{(x+1)^5 \times 5!} + \cdots \right)$

$= \lim_{x \to \infty} \frac{x}{x+1} - \frac{x}{(x+1)^3 \times 3!} + \frac{x}{(x+1)^5 \times 5!} + \cdots + \frac{1}{x+1} - \frac{1}{(x+1)^3 \times 3!} \cdots$

$= \lim_{\infty} \frac{x}{x(1+\frac{1}{x})} - \frac{x}{(x+1)^3 \times 3!} + \cdots$

$= \frac{1}{1} = 1.$

2) (10 points) Find the lowest degree polynomial that will approximate $F(x)$ throughout the given interval with an error of magnitude less than 10^{-3}.

$F(x)=\int_0^x t^2 e^{-t^2}\,dt$, $\quad [0,1]$.

$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \sum_{n=0}^{\infty} \frac{t^n}{n!}$

$e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^8}{3!} + \frac{t^{16}}{4!} + \dots$

$\Rightarrow F x = \int_0^x t^2\left(1 - t^2 + \frac{t^4}{2!} - \frac{t^8}{3!} + \dots\right)$

$= \int_0^x t^2 - t^4 + \frac{t^6}{2!} - \frac{t^{16}}{3!} + \dots$

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$= \left[\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{14} - \frac{t^{17}}{102} + \dots\right]_0^x$

$= \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{14} - \frac{x^{17}}{102} + \frac{x^{19}}{19\times 4!} \cancel{+ \frac{t^{35}}{35\times 5!}}$

For $x = 1$. (At lowest)

$= \frac{1}{3} - \frac{1}{5} + \frac{1}{14} - \frac{1}{102} + \frac{1}{19\times 4!} \left|\frac{1}{4200}\right| < 10^{-3}$

$\Rightarrow F x = \int_0^x t^2 e^{-t^2}\,dt = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{14} - \frac{x^{17}}{102} + \frac{x^{19}}{19\times 4!}$

with errors less than 10^{-3}

5

3) (5 points) Find a Cartesian equation for the curve $r = 4\cos\theta$.

$\boxed{x = r\cos\theta}$

$\Rightarrow r = 4\cos\theta$ (with $\cos\theta = \frac{x}{r}$

$\Rightarrow r = \frac{4x}{r}$ $\qquad r^2 = (r\cos\theta)^2 + (r\sin\theta)^2$

$r^2 = 4x \longrightarrow \quad \Rightarrow x^2 + y^2$.

$\Rightarrow x^2 + y^2 - 4x = 0$.

$x^2 + y^2 - 4x + 2 - 2 = 0$. $\qquad$ radius. $\sqrt{4} = 2$.

$x^2 - 4x + 4 + y^2 = 4$ $\quad \rightarrow \quad$ center $I(2,0)$

$(x-2)^2 + y^2 = 4$.

$(x-2)^2 + y^2 = 4$. $\qquad$ it is a circle centered on $(2,0)$ with r 2

$x^2 + y^2 = 3y$

$x^2 + y^2 - 3y$

$y + \frac{3}{2}$

$2\sin\theta = -3$.

$\theta = -\frac{3}{2}$.

$-\frac{\pi}{2}$

0

$\frac{\pi}{2}$

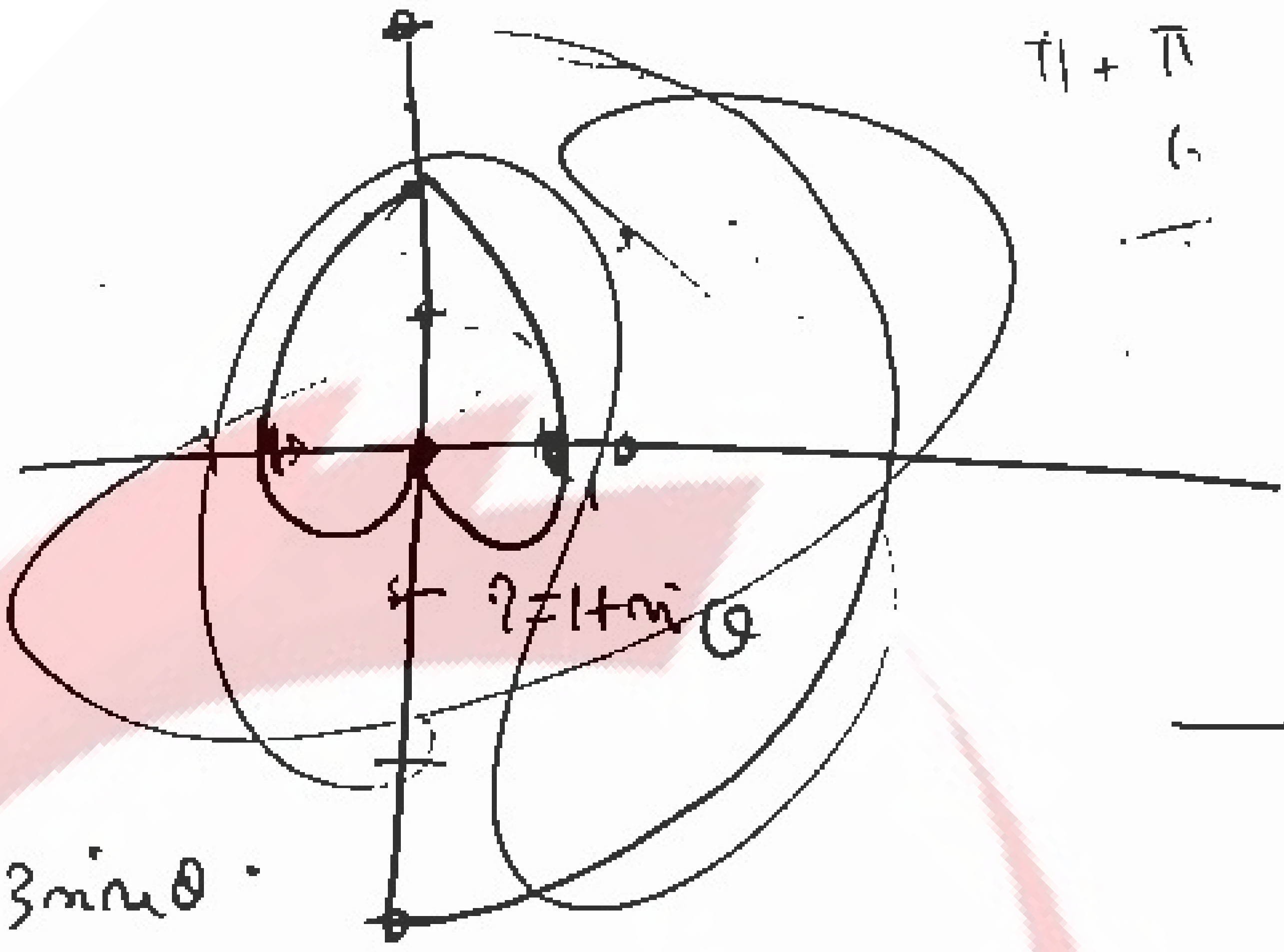
symmetric w.r.t. y-axis.

4) a) **(5 points)** Find all the intersection points of the circle $r = 3\sin\theta$ and the cardioid $r = 1 + \sin\theta$.

b) **(10 points)** Find the area of the region inside the circle $r = 3\sin\theta$ but outside the cardioid $r = 1 + \sin\theta$.

a)

$\pi + \frac{\pi}{6}$



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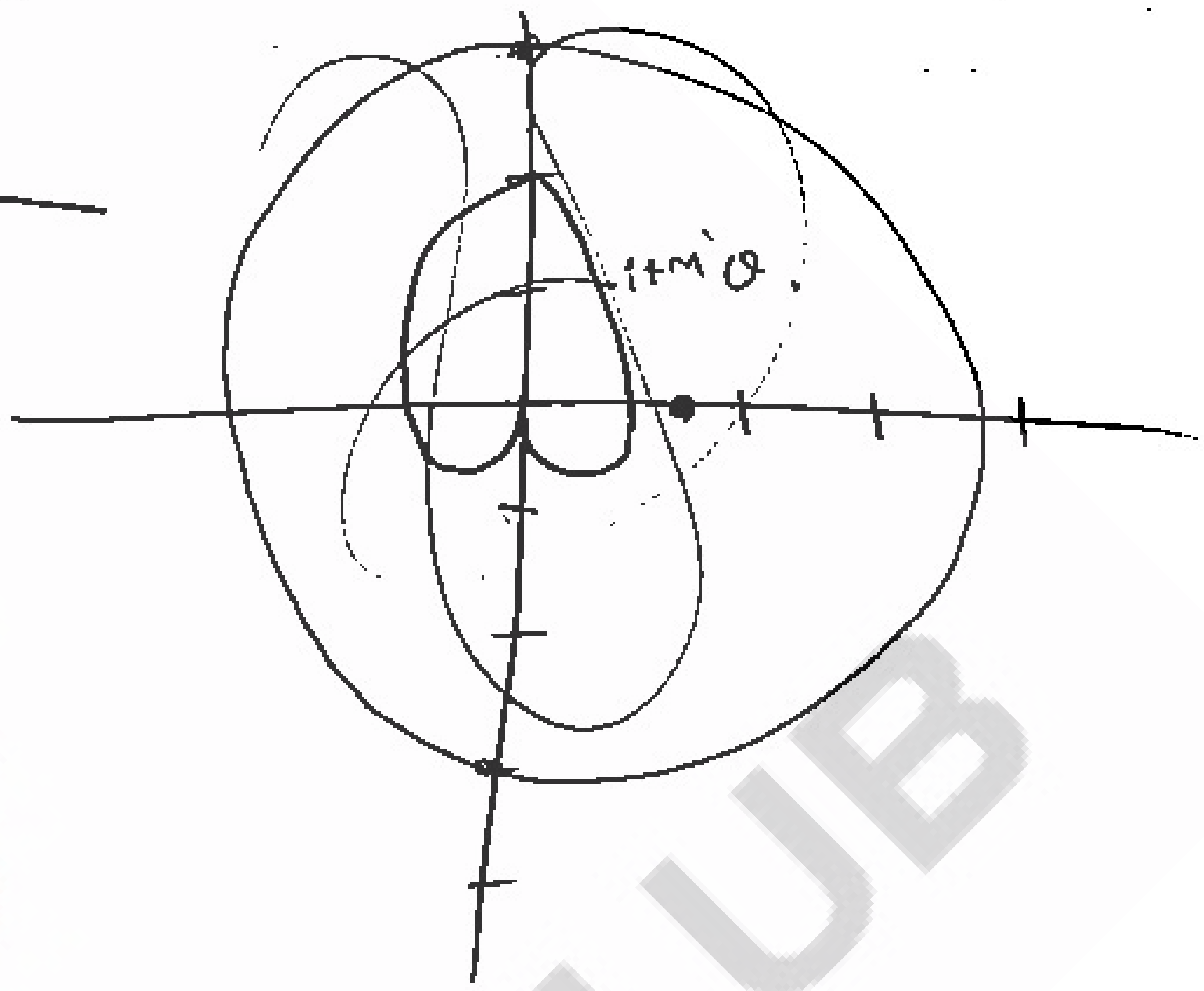
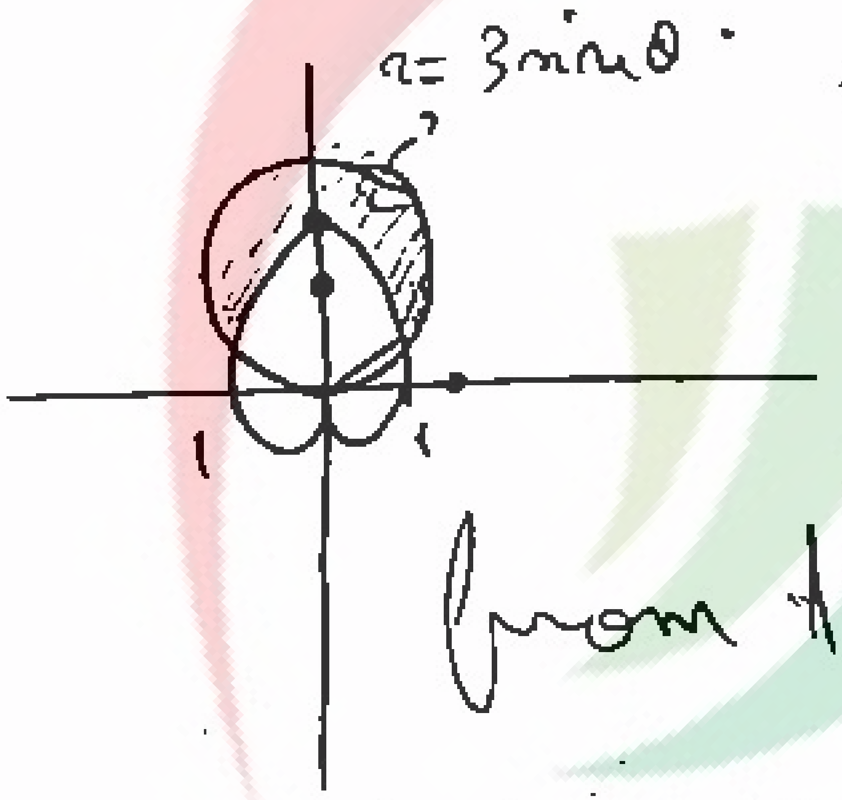




from the graph the point $(0,0)$ or $(0,\pi)$ is a intersection...

$r = r$. $3\sin\theta = 1 + \sin\theta$.

$\sin\theta = \frac{1}{2}$. $\Rightarrow \theta = \frac{\pi}{6}$

or $\theta = 210$

but the mean interval is $(-\frac{\pi}{2}, \frac{\pi}{2})$

$\Rightarrow$ the intersection points are..

$\pi - \frac{\pi}{6}$

$(0,0)$; $(\frac{3}{2}, \frac{\pi}{6})$; $(\frac{3}{2}, \frac{7\pi}{6})$

6 $\frac{5\pi}{6}$

b) the area is: Area of the circle $- \int_{\frac{\pi}{6}}^{\frac{7\pi}{6}} \frac{1}{2}(1+\sin\theta)^2\, d\theta$.

$= \pi \times \left(\frac{3}{2}\right)^2 - \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (1 + \sin^2\theta + 2\sin\theta)\, d\theta$

$= \frac{9\pi}{4} - \frac{1}{2}\int \left(1 + \frac{1-\cos 2\theta}{2} + 2\sin\theta\right) d\theta$

$= \frac{9\pi}{4} - \frac{1}{2}\left[\theta + \frac{1}{2}\theta\right]_{\frac{\pi}{6}}^{\frac{7\pi}{6}} + \frac{1}{4}\left[\theta\right]_{\frac{\pi}{6}}^{\frac{7\pi}{6}} + \frac{1}{8}\left[-\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{7\pi}{6}}$

5) a) **(5 points)** Sketch the curve $r = 2\sin 5\theta$, given that $0 \le \theta \le \dfrac{\pi}{2}$.

b) **(5 points)** What is the shortest length a θ-interval can have and still produce the complete graph of $r = 2\sin 5\theta$? Justify your answer.

c) **(5 points)** Find the area of one leaf.

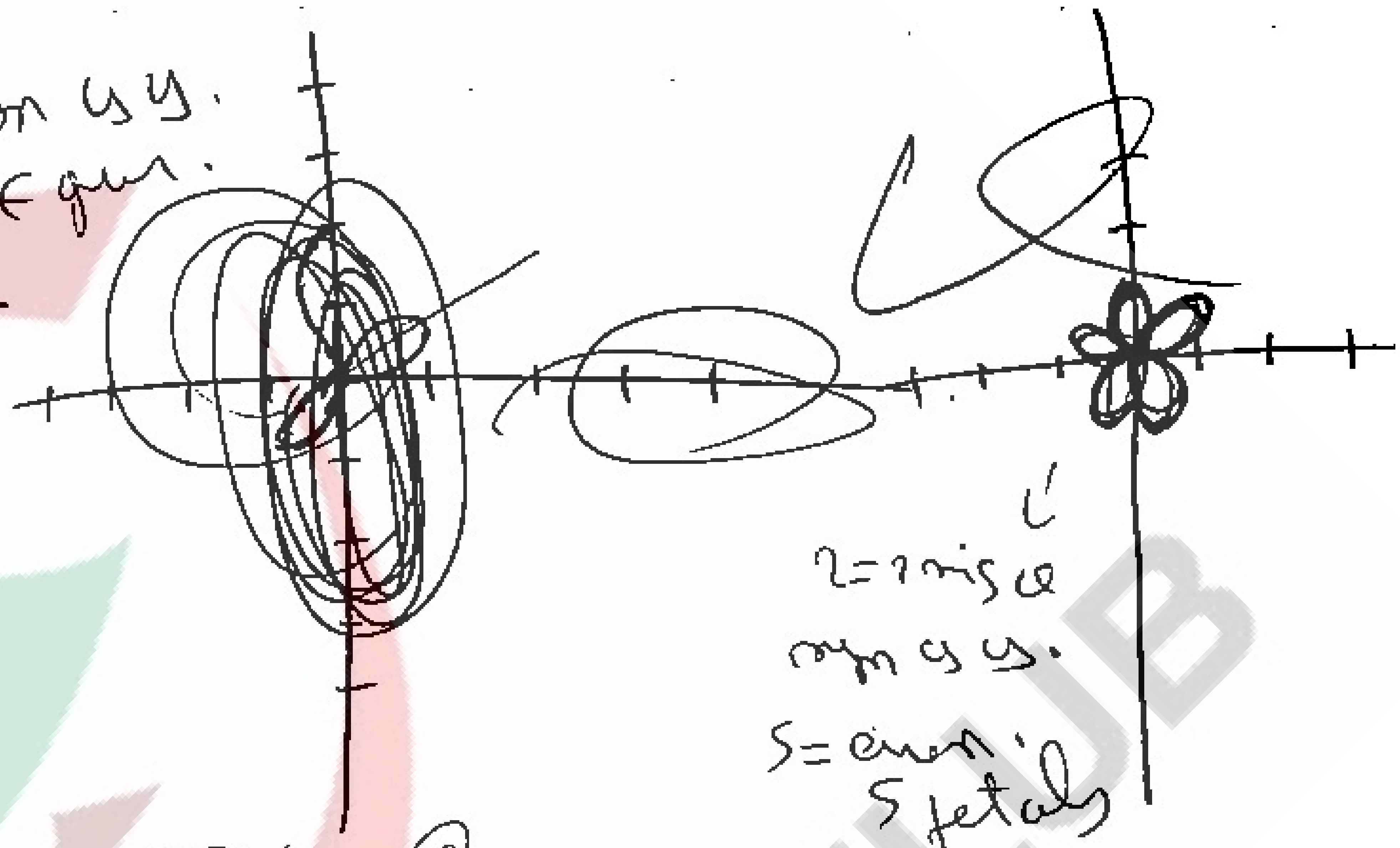
10

$r = 2\sin 5\theta$, $2\sin 5\theta$

sym on y y.

$(-r, -\theta) \in$ graph.
(r, θ)

θ	r
0	0
$\frac{\pi}{6}$	0
$\frac{\pi}{5}$	
$\frac{\pi}{3}$	$-1,7$
$\frac{\pi}{4}$	$-1,4$
$\frac{\pi}{2}$	2



$r = 2\sin 5\theta$
sym y y.
5 = odd
5 petals

b) the shortest interval is $\frac{\pi}{5}$ (~~because~~ . $r = 2\sin 5\theta$ is a leminscate with 5 petals, ($n = 5$ odd $= 5$ petals),
$\Rightarrow$ we do the first one then we complete the graph with using that $r = 2\sin 5\theta$ has 5 petal.
and $r = 2\sin 5\theta$ is simetrique with y axis, because $(-r, -\theta) \in$ of the

c) area $=$

$$2\int_0^{\frac{\pi}{5}} (\sin 5\theta)^2 \, d\theta \qquad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\Rightarrow 2\int_0^{\frac{\pi}{5}} \{1 - \cos 10\theta\} \, d\theta$$

$$= 2\left[\theta\right]_0^{\frac{\pi}{5}} - \frac{1}{10}\left\{\sin 10\theta\right\}_0^{\frac{\pi}{5}} = \left(\frac{2\pi}{5}\right) u^2$$

6) (10 points) Find the length of the cardiod $r = 1 + \cos\theta$.

$$\text{length} = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} (1 + \cos^2\theta + 2\cos\theta + \sin^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2 + 2\cos\theta} \qquad \cos\theta \quad 1 - 2\sin^2\frac{\theta}{2}$$

$$\cancel{= 2\left[\theta + \sin\theta\right]_0^{2\pi}}$$

$$\cancel{= 4\pi},$$

$$= \int_0^{2\pi} \sqrt{4 - 2\sin^2\frac{\theta}{2}}$$

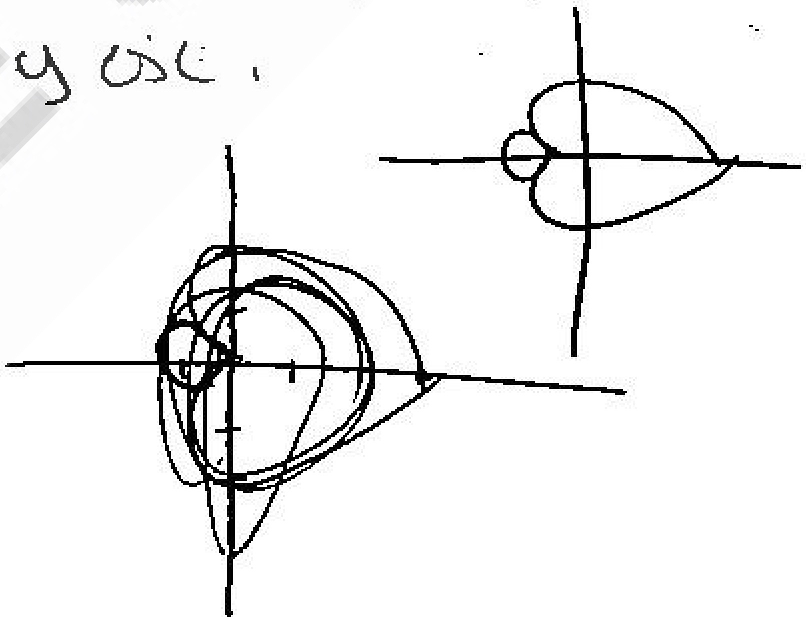
$$= \int_0^{2\pi} \left|\cos\frac{\theta}{2}\right| d\frac{\theta}{2}\, 2\pi$$

$$= 2\int_0^{2\pi} \frac{1}{2}\cos\frac{\theta}{2} = \left[\sin\frac{\theta}{2}\right]_0^{2\pi}$$

5 1

7) (10 points) Let C be the curve $r = 2 + \cos\theta$. Find the equation of the tangent line to the curve C at the point of intersection of C with the positive y-axis.

when $r = 2 + \cos\theta$ intersect the y axis.



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$\Rightarrow r = 2 + \cos\theta$ is a limaçon,

$\Rightarrow$ the intersection points are,

$(2, \frac{\pi}{2})$

$(2, -\frac{\pi}{2})$ negative y axis.

$(0, 0)$.

$$\frac{dy}{d\theta} = ((2+\cos\theta)\sin\theta)'$$

$$\frac{dx}{d\theta} = ((2+\cos\theta)\cos\theta)' \Rightarrow \frac{dy}{dx} = \frac{-\sin\theta^2 + 2\cos\theta + \cos^2\theta}{-\sin\theta\cos\theta - 2\sin\theta - \sin\theta\cos\theta}$$

on $(2, \frac{\pi}{2})$ $\quad \frac{dy}{dx} = \frac{-1}{-2} = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + 2$

8) **(10 points)** Given the points $A(1,-1,0)$, $B(2,1,-1)$ and $C(-1,1,2)$.

a) **(5 points)** Find the area of triangle ABC.

b) **(5 points)** Find an equation of the plane determined by the points A, B and C.

a) area of ABC ?

10 $\quad \vec{AB}\begin{vmatrix}1\\-2\\-1\end{vmatrix} \quad \vec{AC}\begin{vmatrix}-2\\2\\2\end{vmatrix}$

$\frac{\|\vec{AB} \wedge \vec{AC}\|}{2}$ = area of $\widehat{ABC}$

$\vec{AB} \wedge \vec{AC} \Rightarrow \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = i(-6) - j(0) + k(-6)$

$= +6i + 6k$

$\|+6i + 6k\| = \sqrt{36+36} = \sqrt{72}$

$\Rightarrow$ Area of $\widehat{ABC} = \frac{\sqrt{72}}{2}$ ~~$\ldots$~~ $= \frac{\sqrt{72}}{2} = 4{,}24\ u^2$

b) $A, B, C \in$ that plan $\Rightarrow$ this plan have a normal of $\vec{AB} \wedge \vec{AC} = +6i + 6k = \vec{n}$

let $M(x, y, z) \in$ at ABC

$\vec{AM} \cdot \vec{AB} \wedge \vec{AC} = 0$ ($\vec{n}$ normal at A, B, C)

$\vec{AM}\begin{vmatrix}x-1\\y+1\\z\end{vmatrix}$

$(x-1)i + (y+1)j + (z)k \cdot [6i + 6k] = 0$

$\Rightarrow 6x - 6 + 6z = 0$

$\Rightarrow x + z - 1 = 0$ is the eq of ABC plan

9) (10 points) Find the volume of the parallelepiped (Box) determined by

$$\vec{u} = \vec{i} + \vec{j} - 2\vec{k}$$
$$\vec{v} = -\vec{i} - \vec{k}$$
$$\vec{w} = 2\vec{i} + 4\vec{j} - 2\vec{k}$$

$V = |\vec{u} \cdot (\vec{v} \wedge \vec{w})|$.

$\Rightarrow V = \left| \begin{vmatrix} 1 & 1 & -2 \\ -1 & 0 & -1 \\ 2 & 4 & -2 \end{vmatrix} \right| = 1(\times 4) - 1(4) - 2(-4)$

$= 4 - 4 + 8$

$= \underline{8} \, u^3$.

10

10) (10 points) Find a parametrization for the line segment joining the points $P(3,2,5)$ and $Q(1,3,2)$ moving from P to Q.

10

$P, Q \in$ at the line $\Rightarrow$ $\vec{PQ}$ // at the line.

$\vec{PQ} \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ let $M(x, y, z) \in$ at that line.

$\vec{PM} = t\,\vec{PQ}$

$\Rightarrow$ $0 < t < 1$ (moving from $P \to Q$)

$x - 3 = -2t$.

$y - 2 = 1t$.

$z - 5 = -3t$.

$\Rightarrow \begin{cases} x = -2t + 3 \\ y = t - 2 \\ z = -3t + 5 \end{cases}$ $0 < t < 1$ is the parametric of that line