1) (15 points) Find the interval of convergence and the series radius. For which value of x does this series converge absolutely and conditionally?

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-5)^n}{n5^n}$$

$$a_n = \left|(-1)^{n+1}\frac{(x-5)^n}{n5^n}\right| = \frac{(x-5)^n}{n5^n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-5)^{\cancel{n+1}}}{(n+1)\times 5^{n+1}} \times \frac{\cancel{n5^n}}{\cancel{(x-5)^n}}\right| = \left|\frac{(x-5)\times n}{(n+1)\times 5}\right| = \frac{|x-5|}{5} \quad X$$

- $|x-5| < 1 \Rightarrow$ Series converges

$-1 < x-5 < 1 \Rightarrow 4 < x < 6$. $\boxed{\text{Radius} = 1}$

11

- $|x-5| > 1 \Rightarrow$ Series diverges.

- $x = 4$:

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n5^n} = \sum_{n=1}^{\infty}(-1)^{2n+1} \times \frac{1}{n5^n}$$

$$u_n = \frac{1}{n5^n} \Rightarrow \text{AST}: \; u_n > 0$$

$u_n \to 0$ when $n \to \infty$.

$u_n > u_{n+1} \Rightarrow$ Series converges.

- $x = 6$: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(1)^n}{n5^n}$

n^{th} root test:

$$\sqrt[n]{\frac{(1)^n}{n5^n}} = \frac{1}{5}\sqrt[n]{\frac{1}{n}}$$

$$\lim_{n\to\infty}\left(\frac{1}{5}\sqrt[n]{\frac{1}{n}}\right) = \frac{1}{5}\times\sqrt{\frac{1}{\infty}} = \frac{1}{5}\times 0 = 0 < 1. \Rightarrow \text{Series ~~diverges~~}$$

$\Rightarrow$ For $4 < x < 6$ the series absolutely converges.

For $x = 4 \Rightarrow$ the series conditionally converges.

19

2) a. **(10 points)** Derive the MacLaurin series of $f(x) = \dfrac{1}{1+x}$.

b. **(5 points)** Use the series found in part (a) to write series of $\dfrac{1}{1+x^2}$.

c. **(10 points)** Use (b) to approximate $\tan^{-1}(0.1)$ with $|\text{error}| \le 0.0001$.

a- $\dfrac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$ 4

b- $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$ 5

c- $\tan^{-1}(x) = \int \dfrac{1}{1+x^2}$

$\Rightarrow \tan^{-1}(x) = \int (1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots) dx$

$\tan^{-1}(x) = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dfrac{x^{11}}{11} + \cdots$

$|\text{error}| \leq 0{,}0001.$ 10

- $\dfrac{(0{,}1)^3}{3} \geq 0{,}0001$ X
- $\dfrac{(0{,}1)^5}{5} \leq 0{,}0001 \Rightarrow$ So we take the first two terms.

$\Rightarrow \boxed{\tan^{-1}(0{,}1) = 0{,}1 - \dfrac{(0{,}1)^3}{3} = 0{,}10033}$

3) **(10 points)** Use series (**Do Not Use L'Hopital's Rule**) to evaluate $\lim_{x \to 0} \dfrac{x^2}{1-\cos x}$.

$$\lim_{x \to 0} \frac{x^2}{1-\cos x}$$

4) Replace the polar equation by an equivalent Cartesian equation. Then describe the graph:

a) **(5 points)** $r = 3\cos\theta$

$r = 3\cos\theta$.

$r^2 = 3r\cos\theta$

$x = r\cos\theta$
$y = r\sin\theta$ $\Big\} \Rightarrow r^2 = x^2+y^2$.

5

$\Rightarrow x^2+y^2 = 3x$. ; $x^2 - 3x + y^2 = 0$.

$(x - \frac{3}{2})^2 + y^2 - \frac{9}{4} = 0$

$(x - \frac{3}{2})^2 + y^2 = \frac{9}{4}$.

The graph is a circle of Center $O(\frac{3}{2}; 0)$ and of radius $R = \sqrt{\frac{9}{4}} = \frac{3}{2}$.

b) **(5 points)** $r = 4\tan\theta\sec\theta$

$r = 4\tan\theta\sec\theta \Rightarrow \frac{r}{\sec\theta} = 4\tan\theta$.

7

$r \times \cos\theta = 4 \times \frac{\sin\theta}{\cos\theta}$

$r \times \cos\theta = 4 \times \frac{r\sin\theta}{r\cos\theta} = 4 \times \frac{y}{x}$

$x = 4 \times \frac{y}{x} \Rightarrow 4y = x^2 \Rightarrow x^2 = 4y$.

parabola

5) **(12 points)** Find the length of the curve $r = \sin^3\left(\frac{\theta}{3}\right)$ $\quad 0 \le \theta \le \frac{3\pi}{2}$.

$L = \int \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta =$ ✓

$\frac{dr}{d\theta} = \cancel{3\cos^2\theta}\; 3\cos\left(\frac{\theta}{3}\right)\sin^2\left(\frac{\theta}{3}\right) \cdot \times \frac{1}{3}$

6

$\left(\frac{dr}{d\theta}\right)^2 = 9\cos^2\left(\frac{\theta}{3}\right)\sin^4\left(\frac{\theta}{3}\right)$

$r^2 = \sin^6\left(\frac{\theta}{3}\right).$

$\Rightarrow L = \int \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta = \int \sqrt{9\cos^2\left(\frac{\theta}{3}\right)\sin^4\left(\frac{\theta}{3}\right) + \sin^6\left(\frac{\theta}{3}\right)}\, d\theta$

$= 3\int \cancel{\cos\theta}\, \sin^2\left(\frac{\theta}{3}\right)\sqrt{9\cos^2\left(\frac{\theta}{3}\right) + \sin^2\left(\frac{\theta}{3}\right)}\, d\theta.$

6) **(10 points)** Make a table of values to draw the curve: $r = 1 + 2\cos\theta$. Label the x and y intercepts.

At origin: $r = 0 \Rightarrow \cos\theta = -\frac{1}{2}$.

$\theta = \frac{2\pi}{3}$; $\frac{5\pi}{3}$

$\tan\frac{2\pi}{3} = -1{,}73$ $\tan\frac{5\pi}{3} = -1{,}73$

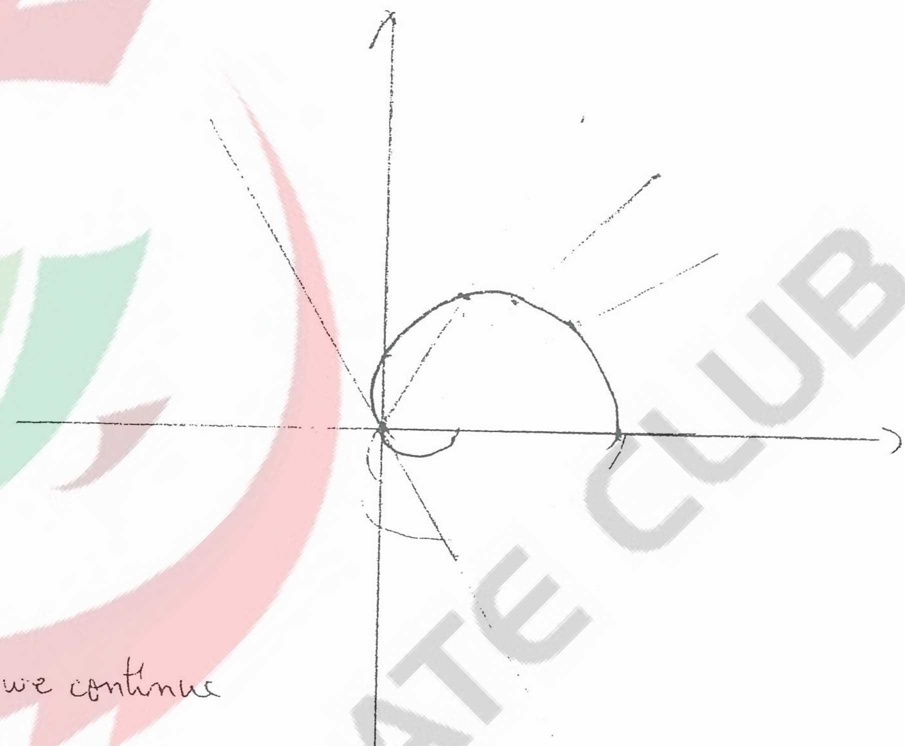
$r = 1 + 2\cos\theta$.

Symmetry:

$\left.\begin{matrix}\theta \to -\theta \\ r \to r\end{matrix}\right\} \Rightarrow$ Symmetry with respect to x-axis.

6

θ	r
0	3.
$\pi/6$	2,73
$\pi/4$	2,41.
$\pi/3$	2.
$\pi/2$	1.
$\frac{2\pi}{3}$	0.
$\downarrow$	$\downarrow$
	π



These values are enough if we continue drawing the curve using the symmetry with respect to Ox.

7) a. **(8 points)** Given the curves $r_1 = (1+\sin\theta)$ and $r_2 = 3\sin\theta$, find their intersection points.

~~$1+\sin\theta$~~ Intersection: $r_1 = r_2$

$\Rightarrow 1+\sin\theta = 3\sin\theta$; $2\sin\theta = 1$; $\sin\theta = \frac{1}{2}$; $\theta = \frac{\pi}{6}$ ✓

$r_1 = 1+\sin\theta$

Symmetry:

$\left.\begin{array}{l}\theta \to \pi - \theta \\ r \to r\end{array}\right\} \Rightarrow$ sym. wrt Oy

θ	r
0	1
$\pi/6$	1,5
$\pi/4$	1,7
$\pi/3$	1,866
$\pi/2$	2
π	0

At origin:

$\sin\theta = -1$

$\Rightarrow \theta = -\frac{\pi}{2}$

$\tan -\frac{\pi}{2} = \infty$

$\Rightarrow$ curve is tangent to y axis

$r_2 = 3\sin\theta$

Symmetry:

$\left.\begin{array}{l}\theta \to -\theta \\ r \to -r\end{array}\right\} \Rightarrow$ Symmetry wrt Oy

θ	r
0	0
$\pi/6$	1,5
$\pi/4$	2,12
$\pi/3$	2,59

θ	r
$\pi/2$	3
π	0

At origin:

$r_2 = 0$

$\sin\theta = 0$

$\theta = 0, \pi, 2\pi$

$\tan 0 = 0$, $\tan\pi = 0$, $\tan 2\pi = 0$

$\Rightarrow$ curve tangent to x axis

b. **(10 points)** Find the area inside r_2 and outside r_1.

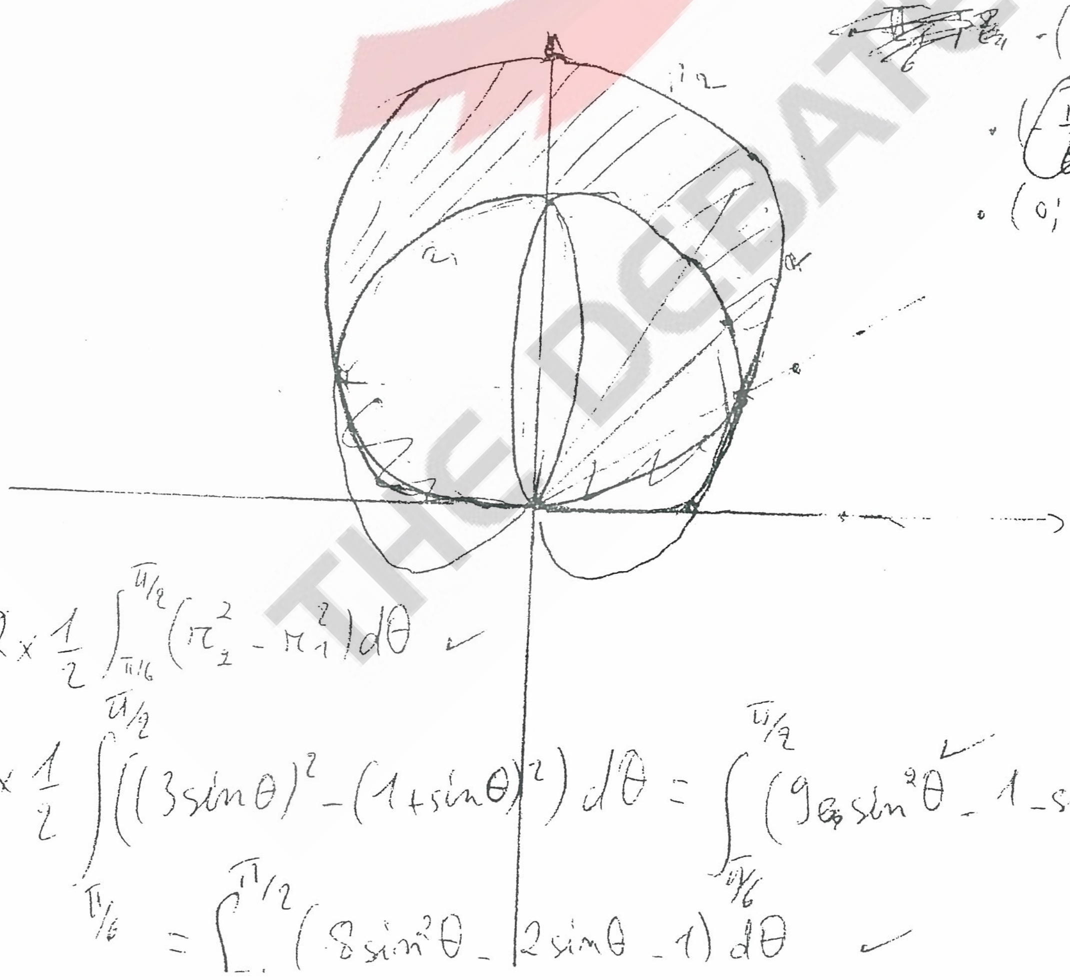
The pts of intersection

$\cdot\left(\frac{\pi}{6} ; 1,5\right)$

$\cdot\left(\frac{\pi}{6} ; 1,5\right)$

$\cdot (0 ; 0)$

3 pts



$a = 2 \times \frac{1}{2}\int_{\pi/6}^{\pi/2} (r_2^2 - r_1^2)\, d\theta$ ✓

$a = 2 \times \frac{1}{2}\int_{\pi/6}^{\pi/2} \left((3\sin\theta)^2 - (1+\sin\theta)^2\right) d\theta = \int_{\pi/6}^{\pi/2} \left(9\sin^2\theta - 1 - \sin^2\theta - 2\sin\theta\right) d\theta$

$= \int^{\pi/2} \left(8\sin^2\theta - 2\sin\theta - 1\right) d\theta$ ✓